Overlap and fractional graph colouring

Thesis

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OVERLAP AND FRACTIONAL GRAPH COLOURING

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Declaration

This thesis describes a variety of techniques for finding overlap colourings of various classes of graphs, tabulates certain values of colouring parameters, and proves a number of general results. The colouring techniques and the tables, and the results for small q in Chapter 5, are entirely my own work, as are the results on equable colourings. The formulation of some of the more general results owes much to the advice of my Supervisor, Dr Holroyd; and the proofs of these results have largely been worked out jointly between us.

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Abstract

Although a considerable body of material exists concerning the colouring of graphs, there is much less on overlap colourings. In this thesis, we investigate the colouring of certain families of graphs. These are the cycle graphs (C_n), the wheel graphs (W_n), the generalized Petersen graphs ($P(p, q)$) and the complete graphs (K_n). There are several results on *equable* colourings (that is, colourings in which all colours occur with equal frequency), but the principal parameters χ_f, $\chi_{r,\lambda}$ and $\chi_f[x]$ are not based on this assumption.

The principal result of Chapter 2 (Theorem 2.1) is that for any positive integer p:

$$\chi_{r,\lambda}(C_{2p+1}) = \max\left\{2r-\lambda, 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil\right\}.$$

This generalizes a theorem of Saul Stahl [23] that for the graph C_n,

$$\chi_r(C_{2p+1}) = 2r + 1 + \left[\frac{r-1}{p}\right] = \chi_r(C_{2p+1}) = 2r + \left\lceil \frac{r}{p} \right\rceil,$$

(in which '[...]' stands for 'the integer part of '...').

In Chapter 3 we find a set of five expressions (Theorem 3.6) for the value of $\chi_{r,\lambda}(W_{2p+2})$ depending on the value of λ.

Chapter 4 discusses overlap colourings and homomorphisms, and introduces a class of graphs which we name *bangles*.

Chapter 5 is concerned with generalized Petersen graphs.We introduce a simple system of symbols to find colourings, and include a study of the fractional chromatic numbers.

Chapter 6 is mainly concerned with the relation between the colouring of complete graphs and the parameters of Design Theory and with codes.

Chapter 7 discusses the representation of the fractional parameters of overlap colourings graphically, making use of a diagram that we call a 'chromatic polygon'.

Chapter 8 is concerned with overlap colourings and statistical applications.

Glossary

For ease of reference, we list here the usage of some symbols and expressions.

n the number of vertices;

r the number of colours at each vertex;

N the total number of colours used in the graph (the 'palette');

f the frequency of occurrence of each colour, if constant. If the frequency is not constant, then we distinguish the frequencies by numerical suffices, $f_1, f_2 \ldots$;

$\chi_r(G)$ the least number of colours required to construct an r-fold colouring;

$\chi_f(G) = \lim_{r \to \infty} \left(\frac{\chi_r(G)}{r} \right)$ the fractional chromatic number of G;

$\chi_{r,\lambda}(G)$ the number of colours required to colour a graph G with r colours per vertex so that the number of colours common to any pair of adjacent vertices is λ.

$\chi_f[x](G) = \inf_{r \to \infty} \left(\frac{\chi_{r,xr}(G)}{r} \right)$, in which r is a multiple of the denominator of x.

There are annexes giving examples of colourings at the end of relevant chapters.

Introduction

Graph colouring theory has of recent years produced several interesting variants.

We recall that the objective of the traditional theory is to find the minimum number of colours required to colour the vertices of a given graph so that adjacent vertices receive distinct colours. The subject of this thesis is one of the variants; we are required to attach *r* colours to each vertex so that any two adjacent vertices share $\boxtimes$ colours. (Once again, and in all the variants discussed below, the objective is to find the minimum number of colours required in order to achieve this.) Though this problem has not received much direct attention, we shall see that it impinges on some aspects of the design of statistical experiments.

Other well-known variants of graph colouring theory are as follows.

Edge and Total Colourings

The 'elements' of a graph comprise its vertices and edges, and also its faces if it is embedded on a surface. An edge, vertex-edge, face-edge, vertex-face, or vertex-face-edge *colouring* of a graph is a colouring of the relevant elements, usually subject to the condition that any two adjacent or *incident* elements must receive distinct colours. There is a large literature on edge colourings, see for example [8]; and on vertex-edge colourings (usually described as *total colourings*); see for example [28]. The other types mentioned have attracted less attention; see for example [26] on edge-face total colourings and [14] on vertex-edge-face colourings. It would clearly be possible to study 'overlap' variants of these. This does not appear to have been done, and this thesis does not study such colourings. Thus we shall not mention total colourings further.

Defective Colourings

Defective colourings were introduced by Cowen, Cowen and Woodall [6]. In this variant, the requirement that no two adjacent vertices share a colour is relaxed. Instead, each is permitted to be adjacent to at most *a certain number* (or sometimes *a certain proportion*) of similarly coloured vertices; or alternatively the total number of same-colour adjacencies in the graph is limited.

There is a certain similarity between the concepts of overlap and defective colourings in that both allow colour overlaps; but it is not easy to see deeper connections, and we have not pursued possible connections in this thesis.

Circular Colourings

In a circular colouring of a graph G, the colours themselves are arranged at equal distances round a circle and two adjacent vertices of G are required to have colours that are *at least a certain distance apart*. Equivalently, each colour occupies a certain colour interval. This gives rise to the concept of *circular*, or *star*, *chromatic number* (see [12], [19], [24], [29]). There does not seem to be a general connection between circular and overlap colourings; however, in the particular case when the graph to be coloured is a cycle, a considerable link between these types of colourings emerges because it is frequently possible to minimize the number of colours required for an overlap colouring, by using circular intervals as the colour sets. Many of the colourings used in Chapter 2 are of this form.

Equable Colourings

The concept of *equitable* vertex-colourings, in which colours have frequencies within 1 of each other, has been studied extensively in the literature (see [5], [15] for example). This thesis sometimes uses the more stringent concept of *equable* colouring, in which all colours occur with equal frequency. As will be seen in Chapter 6, equable colourings of complete graphs are of particular relevance, since they correspond to BIBDs.

Multicolourings

In 1975, Hilton, Rado and Scott [10] introduced the concept of a *multicolouring* of a graph G (or, more generally, a hypergraph); in particular, an *r-fold colouring* allocates a set of r colours to each vertex of G with the requirement that the colour sets at adjacent vertices must be disjoint. Scott's PhD thesis [21] contains an extensive treatment of the basic properties of multicolourings, and in particular calculates all multichromatic numbers of all powers of cycles. The principal interest of [10], however, was in the behaviour of the ratio $\frac{\chi_r(G)}{r}$ for large r (where $\chi_r(G)$ represents the least number of colours required in total to construct an r-fold colouring). They proved (a) an *attainment theorem* and (b) a *periodicity theorem* as follows:

(a) $\lim_{r\to\infty}\left(\frac{\chi_r(G)}{r}\right) = \inf_r\left(\frac{\chi_r(G)}{r}\right)$,

and this limit (the *fractional chromatic number* of G, now denoted by $\chi_f(G)$) is attained for some r_0;

(b) for some positive integer q, the sequence $\{\chi_r(G) - r\chi_f(G) : r > q\}$ is periodic.

The concept of an overlap colouring leads directly from that of a multicolouring; the latter is simply an overlap colouring with the overlap parameter set to 0. Indeed, this is the historical genesis of overlap colouring theory; the paper by Johnson and Holroyd [11] that introduced overlap colourings explicitly generalized the proof methods in [10], as we shall now explain. An important aspect of [10] is their observation that if we replace the problem of allocating *exactly r* colours to each vertex by that of allocating *at least r* colours to each vertex (again with the constraint that adjacent vertices are allocated disjoint sets) then the problem of finding χ_r (for a given r) may be reformulated in terms of *integer programming*, and the problem of finding χ_f is a closely related *linear program*, which is generally easier to solve than an integer program.

Subsequently, Stahl [23] investigated χ_r for various graphs, in particular the cycles and the Kneser graphs [the Kneser graph Kn(p, q)(where $p > 2q$) is the graph whose vertices are the p-subsets of a fixed q-set, two such vertices being adjacent if and only if (as sets) they are disjoint], establishing a number of general theorems and (importantly for this thesis) a complete specification of $\chi_r(C_n)$ for all r, n. The result [5, Theorems 5, 6] is as follows:

$$\chi_r(C_{2p}) = 2r;\ \chi_r(C_{2p+1}) = 2r + \left\lceil \frac{r}{p} \right\rceil = \left\lceil \frac{(2p+1)r}{p} \right\rceil. \tag{1.1}$$

Thus,
$$\chi_f(C_{2p}) = 2;\ \chi_f(C_{2p+1}) = 2 + \frac{1}{p}. \tag{1.2}$$

In 1996, Johnson and Holroyd [11] generalized the multicolouring idea, introducing the concept of an *overlap colouring* of a graph G. In the notation of this thesis, an $[r, \lambda]$-*overlap colouring* of G allocates the colour set S_i to vertex v_i ($0 \le i \le |V(G)| - 1$), such that $|S_i| = r$($0 \le i \le |V(G)| - 1$) and $|S_i \cap S_j| = \lambda$ for each pair i, j such that v_i is adjacent to v_j. They were able to extend the methods of [10] and prove versions of the attainment and periodicity theorems, and other general properties of $[r, \lambda]$-overlap colourings, but proved no precise results for any classes of graphs.

The method of [10] has since been generalized in various directions. Scheinerman and Ullman (1997)[20] give a full account of the process of describing several graph parameters in terms of integer programs, which have linear counterparts that allow for the parameters to be 'fractionalized'.

There have been considerable advances concerning fractional chromatic numbers of graphs, but neither the overlap chromatic number nor its fractional analogue, a version of which was defined in [11], appears to have been further studied (though, as has been mentioned, some work in the theory of statistical designs is relevant, see Chapter 8). The intention of this thesis is to start the systematic study of those parameters.

This thesis discusses the overlap colouring of certain classes of graph, namely the cycle graphs C_n, the wheel graphs W_n, the Petersen and generalized Petersen (GenPet) graphs

(p, q) and the complete graphs K_n. More precisely, it tackles the problem of finding the minimum number of colours needed for an $[r, \lambda]$ colouring of a given graph.

Cycle graph C_n A set of n vertices $v_0, v_1, \ldots, v_{n-1}$, in which v_{i+1} is adjacent to v_i, and v_0 is adjacent to v_{n-1}. We present tables of colour frequencies for some smaller graphs.

Wheel graph W_n A cycle graph C_{n-1}, the 'rim', each vertex $(v_0, \ldots, v_{n-2})$ of which is joined to a further single vertex, the 'hub' (h). This topic is included more particularly because the vertices of these graphs, unlike those of the other three types, are not all similar in situation, the hub having what we may call 'privileged' status.

Petersen and generalized Petersen ('GenPet' graphs $P(p, q)$ Petersen and GenPet graphs consist of two cycles of p vertices, $p = 5$ in the 'classical' Petersen graph, and $p > 5$ in GenPet graphs. One cycle, $v_0, v_1, \ldots, v_{p-1}$, is conveniently displayed as the outer cycle, the remaining p vertices being displayed as an inner cycle, $w_0, w_1, \ldots, w_{p-1}$, each w_i being adjacent to vertex v_i of the outer cycle and to the inner vertices w_{i+q} and w_{i-q}, the suffices being modulo p.

Complete graphs K_n A set of n vertices in which each vertex is adjacent to each other vertex. We show an equivalence between methods of colouring such graphs and the results of Design Theory, and present tables of the same form as, and corresponding precisely with, the tables in the *CRC Handbook of Combinatorial Design* [16]. We adduce a number of simple algorithms for displaying actual colourings.

List Colourings

In a list colouring problem, there is a list of available colours given for each vertex of a graph G, and the problem may allocate different lists to different vertices. The *list chromatic number* of G is then the least l such that the graph can be properly coloured for any allocation of lists, so long as there are at least l colours in each.

The concept does not extend in a simple way to colourings with overlaps, since it would be easy to provide lists for each vertex such that overlap was impossible between certain pairs of adjacent vertices. Thus, it would be necessary to re-state the problem so that there was a list of given length (say l) at each vertex, *and* the lists overlapped by (say) m at any pair of adjacent vertices. Then, for given r, λ, one could ask for the minimum (l, m) such that an $[r, \lambda]$ colouring could be constructed for *any* set of vertex lists of length at least l, overlapping by at least m at each pair of adjacent vertices. This seems to be an interesting area of study, though we do not consider it further in this thesis.

Overlap Colourings and Homomorphisms

In Chapter 4 we consider the place of overlap colourings in the classification of graphs. We note that classifying graphs by overlap chromatic properties is at least as fine a classification as by multichromatic properties, and investigate whether it is equivalent to classifying graphs by their *cores* - the smallest subgraphs to which they have a homomorphism. We show that this is not so; graphs with the same chromatic properties can have non-isomorphic cores.

Overlap Colourings and Codes

If the set of colours used for an overlap colouring is identified with the set of positions in a binary string, then the colour set at each vertex may be regarded as a binary codeword. The set of such codewords is then a *constant-weight binary code*, since each codeword contains the same number of 1s. We refer in particular to [1] and [22]. This connection is explored in Chapter 5.

The Chromatic Polygon

The set of all values $\chi_{r,\lambda}(G)$ for all relevant r, λ is conveniently displayed in a device we refer to as the *chromatic polygon* of G. In Chapter 7 we describe the chromatic polygons of the cycles, wheels and complete graphs and also give some general properties of these polygons.

Overlap Colourings and Statistics

The well-known BIBDs (balanced incomplete block designs) correspond, as we shall see in Chapter 6, to overlap colourings of complete graphs. However, the statistical literature also explores other types of designs, relevant (inter alia) to: experiments in which the different treatments need to be compared with all but one treatment factors held constant; experiments in which the 'closeness' of the treatment factors is relevant. These give rise to the requirement for designs that correspond to overlap colourings of Cartesian products of direct graphs, cycles and so on. The final chapter of this thesis explores this connection.

Chapter 1: General Results

1.1 Basic Definitions

For general graph-theoretic terminology, we refer to [8]; in particular, note that all our graphs are finite and simple.

Let μ be a vertex-colouring of a graph G. We use the following symbols:

$n(G)$ the order of G, i.e., the number of vertices;

$r(G,\mu)$ the number of colours at each vertex;

$\lambda(G,\mu)$ the number of colours common to any pair of adjacent vertices;

$N(G,\mu)$ the total number of colours used by μ;

$f(G,\mu)$ the frequency of occurrence of each colour in μ, if the same for each.

The last four are the *integer parameters* of μ. Usually, the context allows us to drop the arguments G and μ without confusion. We frequently refer to a colouring of a graph with n vertices with the above parameters as an $[r, \lambda, N]$ *colouring*. In Chapter 6 all five parameters are often given, in the order $[n, r, \lambda, N, f]$. The set of all colours used is often called the *palette*.

An *overlap colouring* of a graph G is an assignment of sets of colours to the vertices of G such that each vertex receives the same number of colours and each adjacent pair of vertices shares the same number of colours.

The *overlap chromatic number* of G with parameters r and λ is written as $\chi_{r,\lambda}(G)$, and is the least N for which an $[r, \lambda, N]$ colouring exists. This is a generalization of the concept of *multichromatic number*, χ_r, as defined by Stahl [23]. When only the parameters r, λ are specified, we shall refer to an $[r, \lambda]$ colouring.

In particular contexts, the palette is frequently $[N] = \{1, 2, \dots, N\}$, or an algebraic structure such as the cyclic group of order N, denoted by Z_N. Thus, an $[r, \lambda, N]$ colouring of a graph G is a function μ from the vertices of G to the r-subsets of the palette S, such that $|\mu(v) \cap \mu(w)| = \lambda$ whenever v and w are adjacent vertices. It is sometimes helpful to add a little more structure and envisage the colours at a vertex as arranged in an order, the *site* of a given colour in the vertex being its position in the row; it is convenient to refer to the appearance of a particular colour at a particular site as an *occurrence*. Note that, since we have sets rather than multisets of colours, no colour can occur more than once at any vertex. It is convenient 'shorthand' to refer to an $[r, \lambda]$ colouring of the graph G as '$G[r, \lambda]$'.

Examples of overlap colourings of the cycle graph C_5:

[3, 1] colouring of C_5 (i e $C_5[3, 1]$)

	Colours				
Vertices	1	2	3	4	5
v_0	X	X	X		
v_1			X	X	X
v_2	X	X			X
v_3		X	X	X	
v_4	X			X	X

[3, 2] colouring of C_5

	Colours				
Vertices	1	2	3	4	5
v_0	X	X	X		
v_1		X	X	X	
v_2			X	X	X
v_3	X			X	X
v_4	X	X			X

The two colourings above are *cyclic*, in the following sense: a colouring μ of C_n (with vertices labelled $v_0, \ldots, v_{n-1}$) is *cyclic* if the palette is a cyclic group Z_N and there are a subset S of Z_N and a constant $d \in Z_N$ such that

$$\mu(v_i) = S + id \quad (i = 0, \ldots, n-1) \text{ for some } n.$$

In the above examples, $S = \{1, 2, 3\}$ in both cases and $d = 2, 1$, respectively. This concept extends naturally to colourings of the generalized Petersen graph $P(p, q)$; here there are distinct palettes for the outer and inner vertex cycles.

We describe the colouring of a graph G as *equable* if the frequency f of each of its constituent colours is the same. If G has a non-equable colouring, we sometimes denote the frequencies with which the colours occur by f_i ($i \in \{1,2,\ldots\}$); the number of colours occurring with frequency f_i is denoted by N_i.

When giving colourings explicitly, we normally represent the colour sets of vertices as rows and vertex sets of colours as columns, as in the examples above, but it is occasionally convenient to transpose rows and columns.

1.2 Fundamental Properties of Overlap Colourings

We begin with a fundamental property of the overlap colourings of any non-trivial graph.

Proposition 1.1 Let μ be any overlap colouring of any non-null graph; then

$$N \geq 2r - \lambda.$$

Proof Consider any two adjacent vertices, v and w. Now v has r colours, λ of which occur in w. A further $r - \lambda$ colours are necessary in w to limit the overlap to λ. The total number of colours required is then $2r - \lambda$, and at least this number of colours will be required, that is to say, $N \geq 2r - \lambda$. ■

Our next result is almost as straightforward to prove from first principles; but it is worth going via the concept of a *homomorphism.* Given two graphs G and H, a *homomorphism* from G to H is a function $\theta: V(G) \rightarrow V(H)$ such that $\theta(v)$ is adjacent to $\theta(w)$ whenever v is adjacent to w.

Proposition 1.2 Let $0 \leq \lambda \leq r$. If there is a homomorphism from G to H, then

$$\chi_{r,\lambda}(G) \leq \chi_{r,\lambda}(H).$$

Proof Let θ be a homomorphism from G to H and μ an $[r, \lambda]$ colouring of H; then $\mu \circ \theta$ is an $[r, \lambda]$ colouring of G. ■

Proposition 1.3 If G is any bipartite graph, then $\chi_{r,\lambda}(G) = 2r - \lambda$.

Proof Clearly $\chi_{r,\lambda}(K_2) = 2r - \lambda$; we colour one vertex with $\{1, \ldots, r\}$ and the other with $\{r-\lambda+1, \ldots, 2r-\lambda\}$. Moreover, there is a homomorphism $\theta : G \to K_2$ defined by $\theta(v) = v_1$ or v_2 depending on the partite set to which v belongs. ■

An overlap colouring is *trivial* if it is an equable colouring with $f = 1$ or n.

The total number of occurrences in an equable colouring may be expressed in two ways, either as the total number of colours (N) multiplied by the (constant) frequency (f), or as the number of vertices (n) multiplied by the number of colours at each (r); since these are equal, we have

$$Nf = nr.$$

Proposition 1.4 Let μ be any non-trivial equable overlap colouring of a non-null graph; then:

(i) N and n have a common factor;

(ii) unless $N = n$, N and r have a common factor;

(iii) if n and r are prime, then $r < N = n$.

Proof Since f is an integer, $\frac{nr}{N}$ is an integer. If N and n have no common factor, then N divides r. But it is impossible to colour the graph if $N < r$. Then either $N = r$, in which case each vertex has all the colours, which is trivial, or $N > r$. So N and n have a common factor, and, unless $N = n$, N and r also have a common factor. Moreover, since $N > 1$, if n is prime, N is an integer multiple of n (including the possibility $N = n$). If, in addition, r is prime, then either

(i) $N = nr$, again trivial (but the only colouring when $\lambda = 0$), or

(ii) $N = n$, that is, $r < N = n$. In this case, since both n and r are primes, $r < n - 1$ (with the unique and trivial exception when $n = 3$ and $r = 2$). ■

Corollary 1.5 Let μ be a non-trivial equable overlap colouring of a non-null graph, and suppose that n and r are prime. Then $\lambda \geq 2r - n$; in particular, if $r > \frac{n}{2}$, then $\lambda > 0$.

Proof By Propositions 1.1 and 1.4 (iii), $n \geq 2r - \lambda$; that is, $\lambda \geq 2r - n$. ■

In the particular case where a graph has a *universal vertex* (that is, a vertex that is adjacent to all the other vertices), this places a surprisingly strong restriction on the possibility for equable colourings, which we shall use in Chapters 3 and 6.

Proposition 1.6 Let $n(G) = n$, and suppose that G has a universal vertex; then for any equable colouring we have

$$N = \frac{nr^2}{(n-1)\lambda + r}.$$

Proof Let μ be an equable $[r, \lambda]$ colouring of G, with colour frequency f. Let the colours at the universal vertex be 1, ..., r. The colour 1 also occurs on $f - 1$ other vertices of G; similarly for colours 2, ..., r. Thus there are in total $r(f-1)$ pairs (s_o, s_u) where s_o is an 'other vertex' sharing a colour with a site on the universal vertex s_u. These pairs must account for the $\lambda(n-1)$ colour overlaps between the universal site and the other sites. Thus

$$r(f-1) = \lambda(n-1). \qquad (1)$$

Since μ is equable, $Nf = nr$, so that $f = \frac{nr}{N}$.

Substituting in (1) $\qquad \lambda(n-1) = r\left(\frac{nr}{N} - 1\right),$

which, after rearrangement, gives

$$N = \frac{nr^2}{(n-1)\lambda + r}.$$ ■

1.3 Relationships Between Overlap Colourings of a Graph

Complementary Colourings We define the *complementary colouring* of a colouring μ to be the colouring μ^C in which each vertex v receives exactly those colours that it does not receive in μ. It follows from elementary set theory that the parameters of μ^C are $[N-r,\ N-2r+\lambda,\ N-M]$, where M is the number of colours that occur at every vertex in the original colouring.

Juxtapositions of Colourings Suppose that $\mu_1, \ldots, \mu_m$ are overlap colourings of a graph G, with parameters (n, r_i, λ_i, N_i) $(i = 1, \ldots, m)$ and with disjoint palettes. The *juxtaposition* $\sum_i \mu_i$ is the overlap colouring in which each vertex receives all the colours from the corresponding vertices of each of the colourings μ_i. If we denote the juxtaposition by θ, then clearly

$$\begin{aligned} r(G,\theta) &= \sum_i r_i; \\ \lambda(G,\theta) &= \sum_i \lambda_i; \\ N(G,\theta) &= \sum_i N_i. \end{aligned}$$

Moreover, if each μ_i is equable with frequency f, then θ is equable with frequency f.

Relationships Between Colourings of Related Graphs

Given an overlap colouring μ of a graph H, we can often construct a colouring of a related graph G by finding a homomorphism $\theta: G \to H$ and using the construction in the proof of Proposition 1.2; that is, $\mu \circ \theta$ is an overlap colouring of G with the same parameters $[r, \lambda]$. In particular, in Chapter 2 we shall make extensive use of homomorphisms from C_{2q+1} to C_{2p+1} (where $q > p$). However, these are not the only interesting homomorphisms, even between cycles.

For any integers $p \geq 3$, $c \geq 2$, we may define a homomorphism $\theta: C_{cp} \to C_p$ by 'wrapping' C_{cp}, c times round C_p via the homomorphism $\theta(v_{ap+b}) = v_b$ (where $0 \leq b < p$). This process may be alternatively imagined as *chaining* c copies of C_p together after cutting a link of each 'chain' constituting a p-cycle, so as to produce a path, then joining the ends of the path so produced. The result is to repeat the pattern of an overlap colouring of C_p, c times to produce an overlap colouring of C_{cp}.

In a similar way, we may use an overlap colouring of the generalized Petersen graph $P(p,q)$ and a homomorphism $\theta: P(cp,q) \rightarrow P(p,q)$ to obtain an overlap colouring of $P(cp,q)$. Analogously with the case for cycles, the homomorphism is defined by:

$$\theta(v_{ap+b}) = v_b,\ \theta(w_{ap+b}) = w_b \text{ (where } 0 \le b < p).$$

Again, the process may be imagined as the *chaining* of c copies of $P(p, q)$ to form $P(cp, q)$. We show (Annex 1.1) the result of chaining two copies of $P(7, 3)$ to form $P(14, 3)$. In this case six edges of each copy of $P(p, q)$ are 'severed' and then reconnected in the chaining process, although the way in which the two vertex sets are mapped to $P(cp, q)$ is quite transparent.

1.4 Fractional Parameters

Many graph parameters can be 'fractionalized'.An excellent general account of this topic is given by Scheinerman and Ullman [20]; we shall consider the fractional versions only of chromatic number and overlap chromatic number.

One might intuitively expect that to colour a graph with r colours per vertex would always require r times as many colours as one per vertex (that is, one might expect $\chi_{r,0}(G) = r\chi(G)$). This, however, is not so; to take a very simple example, $\chi(C_5) = 3$, but we need only five colours for a 2-tuple colouring of C_5, so $\chi_{2,0}(C_5) < 2\chi(C_5)$.

For low values of r, $\chi_r(G)$ can vary rather erratically (particularly if G has low symmetry). However, as mentioned in the Introduction, $\lim_{r\to\infty} \frac{\chi_r(G)}{r}$ always exists; moreover, the sequence $\{\chi_r(G) - r\chi_f(G)\}$ eventually settles down and becomes periodic.

Johnson and Holroyd [11] generalized these results to overlap colourings. This thesis examines in detail the 'fractional' version of overlap chromatic numbers of the classes of graphs: cycles, wheels, GenPets and complete graphs.

We now define the *rational parameters* for any overlap colouring μ of a graph G, as follows:

$$x(G,\mu) = \frac{\lambda(G,\mu)}{r(G,\mu)}$$

$$y(G,\mu) = \frac{N(G,\mu)}{r(G,\mu)}.$$

(As for integer parameters, we drop the arguments G and μ where allowed by the context.) The choice of symbols x, y relates to the graphical representation that we describe in Chapter 7.

An $[r,\lambda]$ colouring μ of G is said to be *efficient* if $N(G,\mu) = \chi_{r,\lambda}(G)$; that is, if it is of maximum efficiency over all such colourings of G, where the *efficiency* of μ is defined as

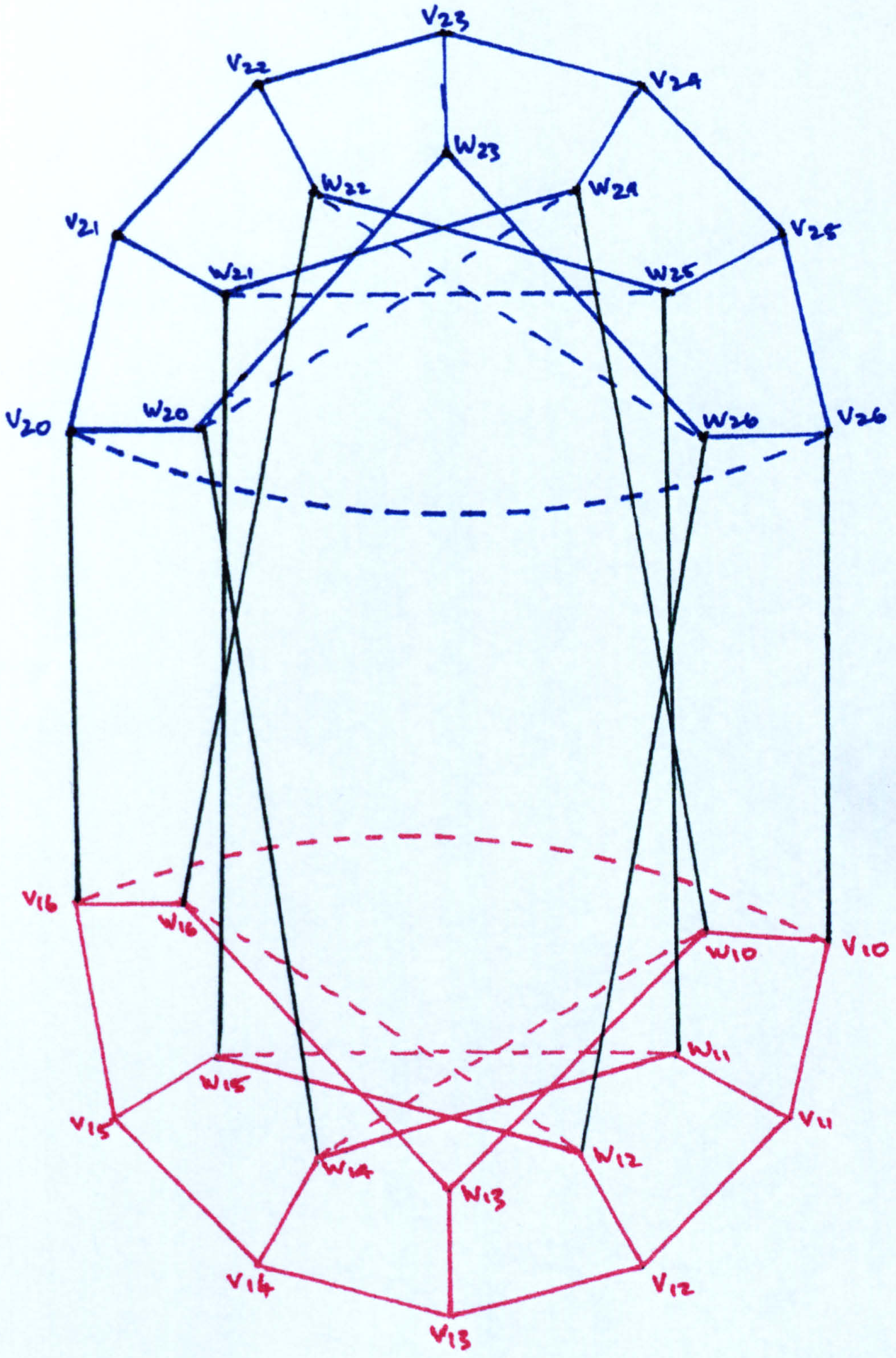
$$\varepsilon(\mu) = \frac{r(G,\mu)}{N(G,\mu)} = \frac{1}{y(G,\mu)}.$$

There are various possible definitions of a 'fractional analogue' of the overlap chromatic number; the one we shall use is as follows:

For each rational number $x \in [0,1]$, we consider the behaviour of $\chi_{r,xr}(G)$ for large r (where r is a multiple of the denominator of x). We denote $\inf_{r\to\infty} \frac{\chi_{r,xr}(G)}{r}$ by $\chi_f[x](G)$; by [10, Theorem 4], this value is attained for some value of r.

In terms of the fractional parameters x and y, $\chi_f[x](G)$ may thus be described as the minimum value of y over all $[r,xr]$ colourings of G.

Annex 1.1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Chapter 2: Cycle Graphs C_n

The Overlap Chromatic Numbers of Cycle Graphs

Since C_{2p} is bipartite, Proposition 1.2 immediately gives $\chi_{r,\lambda}(C_n) = 2r - \lambda$. The case for odd cycles is less trivial; the principal result of this chapter is:

Theorem 2.1 For any positive integer p,

$$\chi_{r,\lambda}(C_{2p+1}) = \max\left\{2r-\lambda, 2(r-\lambda)+\left\lceil \frac{r-\lambda}{p} \right\rceil\right\}.$$

When colouring a cycle using N colours, we use the set $\{1, \dots, N\}$ as the palette, considering these numbers modulo N, that is, as the cyclic group Z_N. The colour set S_0 at vertex v_0 is always $\{1, \dots, r\}$; each colour set S_i is usually an interval $\{x+1, \dots, x+r\}$ modulo N, although we occasionally find it necessary to work modulo N for the first part of the cycle and modulo $N-1$ for the remainder.

Scott [21], Theorem 8, proves the equivalent of our

$$\chi_{r,0}(C_{2p+1}) \le 2r + \left\lceil \frac{r}{p} \right\rceil,$$

and subsequently shows this as a special case of his Theorem 10:

$$\text{for } p \ge 3, \chi_n(C_p^q) \le \left\lceil \frac{np}{\left\lfloor \frac{p}{q+1} \right\rfloor} \right\rceil$$

and proves equality in his Theorem 13.

Colouring Methods

We begin with two general methods.

Method 1 This produces a cyclic colouring with $S = \{1, 2, \dots, r\}$ and $d = r - \lambda$. So, for $i = 0, \dots, n-2$, the overlap is strictly between the last λ colours of v_i and the first λ colours of v_{i+1}; thus,

$$S_i = \{i(r-\lambda)+1, \dots, i(r-\lambda)+r\} = S_0 + i(r-\lambda).$$

This may be divided into two variants:

Method 1(a) The pattern continues from v_{n-1} to v_0, that is, the overlap is strictly between the last λ colours of v_{n-1} and the first λ colours of v_0.

Method 1(b) We still require $|S_{n-1} \cap S_0| = \lambda$, but the overlap is not of the above form. For example, the [3,2] colouring of C_3 with colour sets $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 1\}$ uses Method 1(b).

Proposition 2.2 An n-cycle (either as a graph C_n or as a subgraph of some graph) has a Method 1(a) colouring with parameters $[r, \lambda, N]$ (where $N \geq 2r - \lambda$) if and only if $n(r-\lambda) \equiv 0 \pmod{N}$ or, equivalently, N divides $n(r-\lambda)$.

Proof The colour sets at v_i, v_{i+1} ($i = 0, \ldots, n-2$) clearly overlap by λ colours; we need to verify the conditions under which this is true of the colour sets at v_0 and v_{n-1}. This is so if and only if the last colour of v_{n-1} is the λth colour of v_0; that is,

$$r + (n-1)(r-\lambda) = \lambda.$$

This is clearly equivalent to

$$n(r-\lambda) \equiv 0 \pmod{N}. \qquad \blacksquare$$

If this requirement is not satisfied, then there is no Method 1(a) $[r, \lambda]$ colouring.

Corollary 2.3 If $\lambda = r - 1$ and there is a Method 1(a) colouring then N divides n and so $N = n$.

Moreover,

$$n = N \geq 2r - \lambda = r + 1,$$

so a colouring with $N = n$ is possible if $r \leq n - 1$. $\blacksquare$

A Method 1 colouring is not necessarily equable; an example is C_5:

$$[n, r, \lambda, N] = [5, 7, 1, 15].$$

1	7	13	4	10
2	8	14	5	11
3	9	15	6	12
4	10	1	7	13
5	11	2	8	14
6	12	3	9	15
7	13	4	10	1

Since $nr = 35$ and $N = 15$, there is no integer value of f such that $Nf = nr$. The colours 1, 4, 7, 10, 13 occur three times, the remainder twice.

Method 2 In this method, we colour C_{2q+1} (where $q > p$) as follows. The first $2p + 1$ vertices, $v_0, \ldots, v_{2p}$, are coloured as in Method 1, and then the colour sets alternate, so that $\mu(v_{2p+1}) = \mu(v_{2p-1})$, $\mu(v_{2p+2}) = \mu(v_{2p})$, … , $\mu(v_{2q}) = \mu(v_{2p})$. (Alternatively, let θ be the homomorphism from C_{2q+1} to C_{2p+1} defined by

$$\begin{aligned}\theta(v_i) &= v_i \ (0 \le i \le 2p),\\ &= v_{2p-1} \ (i = 2p+1,\ 2p+3,\ \ldots, 2q-1)\\ &= v_{2p} \quad (i = 2p+2,\ 2p+4,\ \ldots,\ 2q).\end{aligned}$$

Then let $\mu = \mu_0 \circ \theta$ (where μ_0 is the Method 1 colouring of C_{2p+1}).

Proposition 2.4 If $q \ge p$, then $\chi_{r,\lambda}(C_{2q+1}) \le \chi_{r,\lambda}(C_{2p+1})$.

Proof Let μ_0 be any colouring of C_{2p+1} with $\chi_{r,\lambda}(C_{2p+1})$ colours, and use Method 2. ■

We may conveniently include within Method 2 the case $\lambda = 0$, for example, the colouring of C_5 [1, 0]: 1, 2, 3, 2, 3.

In order to determine the conditions under which Method 1 is valid, we introduce the following lemma.

Lemma 2.5 Let $N = 2r - \lambda$ and let S_0, T be the following intervals of Z_N: $S_0 = \{1, \ldots, r\}$, $T = S_0 + t$. Then $|S_0 \cap T| = \lambda$ if and only if $r - \lambda \le t \le r$.

Proof Any two intervals of Z_N of length r must overlap by at least λ.

If $t \le r - \lambda$, then $S_0 \cap T = \{t+1, \ldots, r\}$ and $|S_0 \cap T| = r - t > \lambda$ unless $t = r - \lambda$, in which case $|S_0 \cap T| = \lambda$.

If $r - \lambda < t < r$, let $t = r - \lambda + k$. Note that $k < \lambda$.

Then $T = \{r - \lambda + k + 1, \ldots, 2r - \lambda\} \cup \{1, \ldots, k\}$, and so

$S_0 \cap T = \{r - \lambda + k + 1, \ldots, r\} \cup \{1, \ldots, k\}$, so $|S_0 \cap T| = (\lambda - k) + k = \lambda$.

If $t = r$, then $S_0 \cap T = \{1, \dots, \lambda\}$ and $|S_0 \cap T| = \lambda$.

If $t > r$, then $S_0 \cap T = \{1, \dots, \lambda + t - r\}$ and $|S_0 \cap T| > \lambda$. ■

Proposition 2.6 (i) $\chi_{r,\lambda}(C_3) \le 3(r-\lambda)$ $(0 \le \lambda \le \frac{r}{2})$;

(ii) $\chi_{r,\lambda}(C_3) \le 2r-\lambda$ $(\frac{r}{2} \le \lambda \le r)$.

Moreover, the Method 1 of colouring is valid.

Proof (i) If $0 \le \lambda \le \frac{r}{2}$, then the following Method 1 colouring is valid:

$S_0 \{1, ..., r\}, S_1 = \{r-\lambda+1, ..., r, ..., 2r-\lambda\}$;

$S_2 = \{2(r-\lambda)+1, ..., 3(r-\lambda), 1, ..., \lambda\}$.

Observation If $0 < \lambda < \frac{r}{2}$, then we may produce a colouring in which

$|S_0 \cap S_1| = |S_1 \cap S_2| = \lambda$, and $|S_2 \cap S_0| = \lambda + 1$, using one fewer colours, by setting

$S_2 = \{2(r-\lambda)+1, ..., 3(r-\lambda-1, ..., 1, ..., \lambda+1\}$.

(ii) If $\frac{r}{2} \le \lambda \le r$, then colouring the vertices using Method 1 gives S_0 and S_1 as above, and $S_2 = \{2(r-\lambda)+1, ..., 2r-\lambda, 1, ..., r-\lambda\}$.

Observation In this case, unless $r = \lambda = 1$, we may produce a colouring in which $|S_0 \cap S_1| = |S_1 \cap S_2| = \lambda$ and $|S_2 \cap S_0| = \lambda - 1$, using one extra colour, by setting $S_2 = \{2(r-\lambda)+1, ..., 2r-\lambda+1, 1, ..., r-\lambda-1\}$. ■

These observations will be used in Chapter 3.

We now move to the proof of Theorem 2.1. This requires separate arguments to show that the expression in the theorem is a lower, and that it is an upper, bound.

Stahl [23, Theorem 6] gave a lower-bound proof for non-overlap colourings of C_{2p+1}:

$$\chi_{r,0}(C_{2p+1}) \ge (2r+1) + \lfloor \frac{r-1}{p} \rfloor,$$

or equivalently,

$$\chi_{r,0}(C_{2p+1}) \ge 2r + \left\lceil \frac{r}{p} \right\rceil.$$

Our lower-bound proof generalizes that of Stahl. His crucial step involves obtaining a lower bound on $|S_0 \cap S_{2p}|$ by means of lower bounds on $|S_{2i} \cap S_{2(i+1)}|$

for each i, then equating this to λ, showing that, for fixed r and p and a wide range of values of λ, we can colour C_{2p+1} with the required palette size using Method 1 or Method 2.

Proposition 2.7 $\chi_{r,\lambda}(C_{2p+1}) \geq 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$.

Proof If $2r-\lambda \geq 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$, then the proposition follows from Proposition 1.1.

Assume, then, that $2r-\lambda < 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$, and suppose that μ is an $[r, \lambda]$ colouring of C_{2p+1} using $N = 2r-\lambda+s$ colours.

For $i = 0, \ldots, 2p$, let S_i be the set of colours at v_i.

Consider $S_0 \cap S_2$. Now, S_0 and S_2 must each contain $r-\lambda$ elements disjoint from S_1, and there are only $r-\lambda+s$ such elements available.

Hence, $\quad |S_0 \cap S_2| \geq 2(r-\lambda)-(r-\lambda+s) = r-(\lambda+s)$

Similarly, $\quad |S_2 \cap S_4| \geq r-(\lambda+s)$, and, more generally

$$|S_{2i} \cap S_{2(i+1)}| \geq r-(\lambda+s),\ i = 0, \ldots, p-1.$$

Now, in general, if A, B and C are sets of size r with $|A \cap B| \geq r-\varepsilon_1$ and $|B \cap C| \geq r-\varepsilon_2$, then at most $\varepsilon_1+\varepsilon_2$ of the elements of C can fail to be elements of A, and so

$$|A \cap C| \geq r-(\varepsilon_1+\varepsilon_2);$$

thus $\quad |S_0 \cap S_{2p}| \geq r-p(\lambda+s)$.

But v_0 and v_{2p} are adjacent, and so $|S_0 \cap S_{2p}| = \lambda$.

Then $\quad \lambda \geq r-p(\lambda+s)$,

and so $\quad s \geq \frac{r-\lambda}{p} - \lambda$.

Since s and N are integers, and $N = 2r-\lambda+s$,

it follows that $N \geq 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$

as required. When $\lambda = 0$, this gives $N \geq 2r + \left\lceil \frac{r}{p} \right\rceil$, equivalent to the expression found by Stahl. ■

Our upper-bound proof differs considerably from that of Stahl. Our proof does bear a general similarity to that of Scott, although our approaches differ in detail.

Proposition 2.8 Let $p \geq 1$. Then $\chi_{r,\lambda}(C_{2p+1}) \leq 2r-\lambda$ $(\frac{r}{p+1} \leq \lambda \leq r)$.

Moreover, Method 1 is valid if $\frac{r}{p+1} \leq \lambda \leq \frac{r}{p}$ and Method 2 is valid otherwise.

Proof Let $\frac{r}{p+1} \leq \lambda \leq \frac{r}{p}$, and consider colouring C_{2p+1} using Method 1.

Then v_0 and v_{2p} have the colour sets $S_0 = \{1, \ldots, r\}$ and $S_{2p} = S_0 + 2p(r-\lambda) \pmod{2r-\lambda}$.

Now,

$2p(r-\lambda) = (p-1)(2r-\lambda) + 2r - (p+1)\lambda;$

hence $S_{2p} = S_0 + t$, where $t = 2r-(p+1)\lambda$.

Since $\frac{r}{p+1} \leq \lambda$, then $2r-(p+1)\lambda \leq r$, and since $\frac{r}{p} \geq \lambda$, we have

$2r-(p+1)\lambda = (2r-p\lambda)-\lambda \geq r-\lambda.$

Thus, by Lemma 2.5, the Method 1 colouring is valid.

If $\frac{r}{p+1} \leq \lambda \leq r$, then the Method 1 colouring is valid for C_{2q+1} for some $q < p$,

and therefore a Method 2 colouring is valid for C_{2p+1}. ■

Next, we proceed to the general upper bound argument, including cases where Methods 1 and 2 do not suffice.

Proposition 2.9 $\chi_{r,\lambda}(C_{2p+1}) \leq \max\left\{2r-\lambda, 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil\right\}$.

Proof

Case 1: $\lambda \geq \frac{r}{p+1}$. Since $2r-\lambda$ is an integer, it follows that $2r-\lambda \geq 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$

if and only if $2r-\lambda \geq 2(r-\lambda) + \frac{r-\lambda}{p}$, and this is equivalent to $\lambda \geq \frac{r}{p+1}$.

Thus, by Proposition 2.8, the result follows for $\lambda \geq \frac{r}{p+1}$.

Case 2: $\lambda < \frac{r}{p+1}$. We now show that if $\lambda < \frac{r}{p+1}$, then $\chi_{r,\lambda}(C_{2p+1}) \leq 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$.

In this case, let $N = 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil$.

Let $r-\lambda = ap-y$, where $0 \le y < p$. Thus $\left\lceil \frac{r-\lambda}{p} \right\rceil = a$.

Then

$$pN = 2p(r-\lambda)+p\left\lceil \frac{r-\lambda}{p} \right\rceil = 2p(r-\lambda)+(r-\lambda)+y = (2p+1)(r-\lambda)+y.$$

We now divide the sites of each vertex; the first $r-\lambda$ sites are the *initial sites*, and the remaining λ sites are the *overlap sites*. (Thus, the overlap sites of each vertex are those whose colours overlap with those of the following vertex of the cycle.)

Now, the number of initial sites is $(2p+1)(r-\lambda) = pN-y = (p-y)N+y(N-1)$.

We now colour the initial sites by consecutively allocating colours, using $(p-y)$ cycles with the colours $1, \dots, N$, followed by y cycles using the colours $1, \dots, N-1$. Finally, the overlap sites of each vertex are the first λ colours of the succeeding vertex, and we have constructed a Method 1 colouring (except that we have cycled through $1, \dots, N$ for part of the process and through $1, \dots, N-1$ for the remainder).

This process clearly allocates r colours to each vertex, and gives an overlap of at least λ between adjacent vertices. We need now to verify that the only overlaps between adjacent vertices are between the overlap sites of v_i and the initial sites of v_{i+1}. Note that our assumption $\lambda < \frac{r}{p+1}$ implies $N > 2r-\lambda$, and thus $N-1 \ge 2r-\lambda$. Then, even when cycling through the colours $1, \dots, N-1$, there are no overlaps between the initial sites of adjacent vertices. Thus,

$$\chi_{r,\lambda} \le \max\left\{2r-\lambda, 2(r-\lambda)+\left\lceil \frac{r-\lambda}{p} \right\rceil\right\} \qquad \blacksquare$$

Proof of Theorem 2.1

The theorem now follows from Propositions 2.7 and 2.9. ■

Annex 2.1 deals with minimum palette sizes of certain colourings of the cycle graphs $C_3, C_5, \dots, C_{11}$. In particular, for $1 \le r \le 9$ and $0 \le \lambda \le r-1$, the table gives minimum palette sizes for equable colourings (where they exist) in red, along with the corresponding value of f in blue. Where *either* an equable colouring does not exist *or* there is no equable colouring with $\chi_{r,\lambda}(C_n)$ colours, the value of $\chi_f(C_n)$ is given in violet.

Primitives and Juxtapositions

In Chapter 4 we explore the properties of a wider class of graphs than cycles and show that they nevertheless have the same overlap chromatic numbers as their smallest subgraphs that are odd cycles. In order to do this, it is useful to show that any efficient colouring of C_{2p+1} can be expressed as a juxtaposition of $[r,\lambda]$ colourings with $r \le p$ and $\lambda \le 1$. We call such juxtapositions *additive* colourings.

Let $1 \le r \le p, 0 \le \lambda \le 1$. We denote by $\pi[r,0]$ the efficient $[r,0]$ colouring of C_{2p+1} with $2r+1$ colours, by Method 1 if $r = p$ and by Method 2 if $r < p$; and by $\pi[r+1,1]$ the complement of $\pi[r,0]$ (which by Chapter 1 is indeed an $[r + 1, 1]$ colouring, since no colour occurs at every vertex). Finally, we denote by $\pi[1,1]$ the colouring that allocates the same colour at each vertex. These will be called the *primitive* colourings.

Since juxtapositions are defined only for colourings with disjoint palettes, when we define juxtapositions of primitive colourings we always translate the palettes of the primitives so that their palettes are disjoint. For example, $\pi[r,0]+\pi[r+1,1]$ will refer to the juxtaposition of

$\pi[r,0]$ (using palette $\{1, 2, ..., 2r + 1\}$)

with $\pi[r+1,1]$ (using the translated palette $\{2r+2, 2r+2, ..., 4r+2\}$).

Theorem 2.10 Let $r-\lambda = np+i$ (where $1 \le i \le p$). The following $[r,\lambda]$ colouring μ of C_{2p+1} is efficient:

If $\lambda \le n$, (so that $\chi_{r,\lambda}(C_{2p+1}) = 2(r-\lambda)+\left\lceil \frac{r-\lambda}{p} \right\rceil$), then

$\mu = \pi[1,0]+(n-1)\pi[p,0]+\lambda\pi[p+1,1]$;

if $\lambda > n$, then $\mu = \pi[i+1,1]+(\lambda-n-1)\pi[1,1]+n\pi[p+1,1]$ (so that $\chi_{r,\lambda}(C_{2p+1}) = 2r-\lambda$).

Proof Case 1: If $\lambda \le n$, then the palette size of $\pi[i,0]+(n-\lambda)\pi[p,0]+n\pi[p+1,1]$

is $2i+1+(n-\lambda)(2p+1)+\lambda(2p+1)=2(np+i)+n+1=2(r-\lambda)+\left\lceil \frac{r-\lambda}{p} \right\rceil$, as required.

Case 2: If $\lambda > n$, then the palette size of $\pi[i+1,1]+(\lambda-n-1)\pi[1,1]+n\pi[p+1]$

is $2i+1+(\lambda-n-1)+n(2p+1)=2(np+i)+\lambda=2(r-\lambda)+\lambda=2r-\lambda$, as required.

■

Annex 2.2 shows primitive (emboldened) and additive colourings of a range of cycle graphs. We arrange them as in Annex 2.3 in blocks $Q_{\lambda,n}$ $(\lambda, n \ge 0)$ as follows.

The rows of $Q_{\lambda,n}$ are indexed by λ and the columns by n, where the block $Q_{\lambda,n}$ contains the $[np+i+\lambda,\ \lambda]$ colourings ($1 \le i \le p$); we denote by $Q_{\lambda,n}(i)$ the ith element of the block.

In row 0, the block $Q_{0,0}$ has the primitive colourings $Q_{0,0}(i) = \pi[i,0]$ $(i=1,\dots,p)$.

In row 1, the block $Q_{1,0}$ has the primitive colourings $Q_{1,0}(i) = \pi[i+1,1]$ $(i=1,\dots,p)$.

Now let π be some primitive colouring. We use the notation $Q_{\lambda,n}+\pi$ to represent the set of colourings $Q_{\lambda,n}(i)+\pi$ $(1 \le i \le p)$. The remaining blocks of colourings are now constructed iteratively from $Q_{0,0}$ and $Q_{1,0}$ and the primitives $\pi[1,1]$, $\pi[p,0]$ and $\pi[p+1,1]$ as follows.

In row 0, we set $Q_{0,n}=Q_{0,n-1}+\pi[p,0]$ $(n \ge 1)$, so that

$$Q_{0,n}=Q_{0,0}+n\pi[p,0] \quad (n \ge 1).$$

In column 0, we set $Q_{\lambda,0}=Q_{\lambda-1,0}+\pi[1,1]$ $(\lambda \ge 2)$, so that

$$Q_{\lambda,0}=Q_{1,0}+(\lambda-1)\pi[1,1] \quad (\lambda \ge 2).$$

When $\lambda, n>0$, we set $Q_{\lambda,n}=Q_{k-1,n-1}+\pi[p+1,1]$. Thus,

if $\lambda \le n$, then $Q_{\lambda,n}=Q_{0,n-\lambda}+\lambda\pi[p+1,1] = Q_{0,0}+(n-\lambda)\pi[p,0]+\lambda\pi[p+1,1]$;

if $\lambda > n$, then $Q_{\lambda,n}=Q_{\lambda-n,0}+n\pi[p+1,1]=Q_{1,0}+(\lambda-n-1)\pi[1,1]+n\pi[p+1,1]$.

In Annex 2.3 we show part of the table for C_9 and a general table.

Annex 2.1

Equable and Non-Equable Colourings of C_n

		C_3			C_5			C_6			C_7			C_8			C_9			C_{10}			C_{11}		
r	λ	N	f	N	N	f	N	N	f	N	N	f	N	N	f	N	N	f	N	N	f	N	N	f	N
1	0	3	1		5	1	3	2	3		7	1	3	2	4		3	3		2	5		11	1	3
2	0	6	1		5	2		4	3		7	2	5	4	4		6	3	5	4	5		11	2	5
2	1	3	2		5	2	3	3	4		7	2	3	4	4	3	3	6		5	4	3	11	2	3
3	0	9	1		15	1	8	6	3		7	3		6	4		9	3	7	6	5		11	3	7
3	1			6	5	3		6	3	5	7	3	5	6	4	5	9	3	5	5	6		11	3	5
3	2			4	5	3	4	6	3	4	7	3	4	4	6		9	3	4	5	6	4	11	3	4
4	0	12	1		10	2		8	3		14	2	10	8	4		9	4		8	5		11	4	9
4	1			9	10	2	8	12	2	7	7	4		8	4	7	9	4	7	10	4	7	11	4	7
4	2	6	2		10	2	6	6	4		7	4	6	8	4	6	6	6		10	4	6	11	4	6
4	3			5	5	4		6	4	5	7	4	5	8	4	5	9	4	5	5	8		11	4	5
5	0	15	1		25	1	13	10	3		35	1	12	10	4		15	3	12	10	5		11	5	
5	1			12			10	10	3	9			10	10	4	9	9	5		10	5	9	11	5	9
5	2			9			8	10	3	8			8	8	5		9	5	8	10	5	8	11	5	8
5	3			7			7			7	7	5		8	5	7	9	5	7	10	5	7	11	5	7
5	4			6			6	6	5		7	5	6	8	5	6	9	5	6	10	5	6	11	5	6
6	0	18	1		15	2		12	3		14	3		12	4		18	3	14	12	5		22	3	14
6	1			15	15	2	13	12	3	11	14	3	12	12	4	11	18	3	12	12	5	11	11	6	
6	2			12	10	3		12	3	10	14	3	10	12	4	10	18	3	10	10	6		11	6	10
6	3	9	2		10	3	9	9	4		14	3	9	12	4	9	9	6		10	6	9	11	6	9
6	4			8	10	3	8	12	3	8	14	3	8	8	6		9	6	8	10	6	8	11	6	8
6	5			7			7			7	7	6		8	6	7	9	6	7	10	6	7	11	6	7
7	0	21	1		35	1	18	14	3		49	1	17	14	4		21	3	16	14	5		77	1	16
7	1			18			15	14	3	13			14	14	4	13	21	3	14	14	5	13			14
7	2			15			13	14	3	12			12	14	4	12	21	3	12	14	5	12			12
7	3			12			11	14	3	11			11	14	4	11	21	3	11	14	5	11	11	7	
7	4			10			10	14	3	10			10	14	4	10	21	3	10	10	7		11	7	10
7	5			9			9			9			9			9	9	7		10	7	9	11	7	9
7	6			8			8			8			8	8	7		9	7	8	10	7	8	11	7	8
8	0	24	1		20	2		16	3		28	2	19	16	4		18	4		16	5		22	4	18
8	1			21	20	2	18	16	3	15			17	16	4	15	18	4	16	16	5	15	22	4	16
8	2			18	20	2	15	16	3	14	14	4		16	4	14	18	4	14	16	5	14	22	4	14
8	3			15	20	2	13	16	3	13	14	4	13	16	4	13	18	4	13	16	5	13	22	4	13
8	4	12	2		20	2	12	12	4		14	4	12	16	4	12	12	6		16	5	12	22	4	12
8	5			11			11	12	4	11	14	4	11	16	4	11	12	6	11			11	11	8	
8	6			10	10	4		12	4	10	14	4	10	16	4	10	12	6	10	10	8		11	8	10
8	7			9			9			9			9			9	9	8		10	8	9	11	8	9
9	0	27	1		45	1	23	18	3		21	3		18	4		27	3	21	18	5		33	3	20
9	1			24			20	18	3	17	21	3	19	18	4	17	27	3	18	18	5	17	33	3	18
9	2			21			18	18	3	16	21	3	17	18	4	16	27	3	16	18	5	16	33	3	16
9	3			18	15	3		18	3	15	21	3	15	18	4	15	27	3	15	15	6		33	3	15
9	4			15	15	3	14	18	3	14	21	3	14	18	4	14	27	3	14	15	6	14	33	3	14
9	5			13	15	3	13	18	3	13	21	3	13	18	4	13	27	3	13	15	6	13	33	3	13
9	6			12	15	3	12	18	3	12	21	3	12	12	6		27	3	12	15	6	12	33	3	12
9	7			11			11			11			11			11			11	15	6	11	11	9	
9	8			10			10			10			10			10			10	10	9		11	9	10

Annex 2.2

Primitive and Additive Colourings of C_{2p+1}

		C_3		C_5		C_7		C_9		C_{11}		C_{13}		C_{15}		C_{17}		C_{19}	
r	λ	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*	*Ne*	*Nn*
1	0	3			3		3	3			3		3	3			3		3
2	0	6		5			5		5		5		5	5			5		5
2	1	3			3		3	3			3		3	3			3		3
3	0	9			8	7			7		7		7		7		7		7
3	1		6	5			5		5		5		5		5		5		5
3	2		4		4		4		4		4		4		4		4		4
4	0	12		10			10	9			9		9		9		9		9
4	1		9		8	7			7		7		7		7		7		7
4	2	6			6		6	6			6		6		6		6		6
4	3		5	5			5		5		5		5		5		5		5
5	0	15			13		12		12	11			11		11		11		11
5	1		12		10		10	9			9		9		9		9		9
5	2		9		8		8		8		8		8		8		8		8
5	3		7		7	7			7		7		7		7		7		7
5	4		6		6		6		6		6		6		6		6		6
6	0	18		15		14			14		14	13			13		13		13
6	1		15		13		12		12	11			11		11		11		11
6	2		12	10			10		10		10		10	10			10		10
6	3	9			9		9	9			9		9	9			9		9
6	4		8		8		8		8		8		8		8		8		8
6	5		7		7	7			7		7		7		7		7		7
7	0	21			18		17		16		16		16	15			15		15
7	1		18		15		14		14		14	13			13		13		13
7	2		15		13		12		12		12		12		12		12		12
7	3		12		11		11		11	11			11		11		11		11
7	4		10		10		10		10		10		10		10		10		10
7	5		9		9		9	9			9		9		9		9		9
7	6		8		8		8		8		8		8		8		8		8
8	0	24		20			19	18			18		18		18	17			17
8	1		21		18		17		16		16		16	15			15		15
8	2		18		15	14			14		14		14		14		14		14
8	3		15		13		13		13		13	13			13		13		13
8	4	12			12		12	12			12		12		12		12		12
8	5		11		11		11		11	11			11		11		11		11
8	6		10	10			10		10		10		10		10		10		10
8	7		9		9		9	9			9		9		9		9		9
9	0	27			23	21			21		20		20		20		20	19	
9	1		24		20		19		18		18		18		18	17			17
9	2		21		18		17		16		16		16		16		16		16
9	3		18	15			15		15		15		15	15			15		15
9	4		15		14		14		14		14		14		14		14		14
9	5		13		13		13		13		13		13		13		13		13
9	6		12		12		12		12		12		12		12		12		12
9	7		11		11		11		11	11			11		11		11		11
9	8		10		10		10		10		10		10		10		10		10

Ne = number of colours in an equable colouring *Nn* = number of colours in a non-equable colouring

Primitives are emboldened

Annex 2.3

Primitive/Additive Tables

C_9 (As example)

Q₁				Q₂				Q₃				
[1, 0] (3)	**[2, 0] (5)**	**[3, 0] (7)**	**[4, 0] (9)**	[5, 0] (12)	[6, 0] (14)	[7, 0] (16)	[8, 0] (18)	[9, 0] (21)	[10, 0] (23)	[11, 0] (25)	[12, 0] (27)	[13, 0] (30)
[2, 1] (3)	**[3, 1] (5)**	**[4, 1] (7)**	**[5, 1] (9)**	[6, 1] (12)	[7, 1] (14)	[8, 1] (16)	[9, 1] (18)	[10, 1] (21)	[11, 1] (23)	[12, 1] (25)	[13, 1] (27)	[14, 1] (30)
[3, 2] (4)	[4, 2] (6)	[5, 2] (8)	[6, 2] (10)	[7, 2] (12)	[8, 2] (14)	[9, 2] (16)	[10, 2] (18)	[11, 2] (21)	[12, 2] (23)	[13, 2] (25)	[14, 2] (27)	[15, 2] (30)
[4, 3] (5)	[5, 3] (7)	[6, 3] (9)	[7, 3] (11)	[8, 3] (13)	[9, 3] (15)	[10, 3] (17)	[11, 3] (19)	[12, 3] (21)	[13, 3] (23)	[14, 3] (25)	[15, 3] (27)	[16, 3] (30)
[5, 4] (6)	[6, 4] (8)	[7, 4] (10)	[8, 4] (12)	[9, 4] (14)	[10, 4] (16)	[11, 4] (18)	[12, 4] (20)	[13, 4] (22)	[14, 4] (24)	[15, 4] (26)	[16,4] (28)	[17, 4] (30)
[6, 5] (7)												[18, 5] (31)

C_{2p+1}

λ	Q₁	Q₂		Q$_n$	Q$_{n+1}$
0	**[1, 0]** .... **[p, 0]**	| [p + 1, 0] ...... [2p, 0]	| .... [(n - 1)p + 1, 0]	[np, 0] | [np + 1, 0]	[(n + 1)p, 0] | ...
1	**[2, 1]** ... **[p + 1, 1]**	| [p + 2, 1] ...... [2p + 1,1]	| .... | [(n - 1)p + 2, 1]	[np + 1, 1] | [np + 2, 1]	[(n + 1)p + 1, 1] |
...					
....					
λ	[λ + 1, λ] ... [p + λ] | [p + λ + 1] ... [2p +λ, λ] .|		.... | [(n - 1)p + λ + 1, λ]	[np + λ, λ] [(np +λ+ 1, λ]	[(n + 1)p +λ, λ] | ...
λ + 1			| [(n - 1)p + λ + 2, λ + 1] ...	[np + λ + 1, λ + 1] | [(np + λ + 2, λ + 1] ...	[(n + 1)p +λ + 1, λ + 1] | ...

....

Emboldening indicates a primitive, to which we add **[1, 1]**.

Chapter 3: Wheel Graphs W_n

A *wheel graph* W_n consists of a cycle graph C_{n-1} which we shall for convenience call the *rim*, each vertex of which is adjacent to a further single vertex, which we shall call the *hub*. We label the rim vertices $v_0, v_1, \ldots, v_{n-2}$, as in Chapter 2, and the hub vertex h.

Since W_n is a subgraph of K_n, there is an equable $[r, \lambda]$ colouring of W_n if there exists an equable $[r, \lambda]$ colouring of K_n.

Equable colourings

Since the hub vertex is a universal vertex, we have the following restriction on equable colourings of wheels.

Proposition 3.1 If there exists an equable $[r, \lambda]$ colouring of the wheel graph W_n, then

$$N = \frac{nr^2}{(n-1)\lambda + r}.$$

Proof This is an immediate consequence of Proposition 1.6. ■

In the remainder of this chapter, we do not assume equability. We discuss separately W_{2q+1} and W_{2q+2}.

General Colourings

Proposition 3.2 For any wheel graph W_{2q+1},

$$\chi_{r,\lambda}(W_{2q+1}) = \max\{2r-\lambda, 3(r-\lambda)\}.$$

Proof Any component triangle consisting of two adjacent rim vertices and the hub is a C_3 graph, which can be minimally coloured as such. Moreover, if the colour sets at the vertices v_0, v_1, h are A, B, C respectively, then the remaining rim vertices of any W_{2q+1} can be coloured alternately with A and B, as in Method 2 of Chapter 2.

We have seen that for C_{2p+1}, $N = \max\left\{2r-\lambda, 2(r-\lambda) + \left\lceil \frac{r-\lambda}{p} \right\rceil\right\}$.

In the particular case of C_3, $p = 1$, so that

$$\chi_{r,\lambda}(W_{2p+1}) = N = \max\{2r-\lambda, 2(r-\lambda)+\lceil r-\lambda \rceil\} = \max\{2r-\lambda, 3(r-\lambda)\}. \quad \blacksquare$$

We then consider colouring the graph W_{2p+2}. Since an even wheel has an odd number of rim vertices, we cannot argue as above. Let

μ be an $[r, \lambda]$ colouring of an even wheel graph W_{2p+2};

S be the set of all colours used;

$H = \mu(h)$ be the set of colours at the hub;

θ be the colouring of C_{2p+1} using H; that is, the *hub component* of the colouring of W_{2p+2}.;

ω the colouring of C_{2p+1} using $S-H$, that is, the *rim component* of that colouring;

so that $\mu' = \theta + \omega$, where μ' is the restriction of μ to C_{2p+1}.

Thus, θ and ω use λ and $r-\lambda$ colours per rim vertex, respectively. Individually, these are not necessarily constant-overlap colourings, although between adjacent vertices the sum of the overlaps is always λ. (However, as we shall see, we need only consider cases where θ and ω each take just two consecutive overlap values.)

Since the number of colours in θ is fixed as r, in order to find $\chi_{r,\lambda}(W_{2p+2})$ we need to minimize the number of colours in ω. Proposition 3.3 (below) shows that we minimize the number of colours required by maximizing the mean or minimum overlap. In order to maximize the overlaps in ω, we need correspondingly to minimize the overlaps in θ.

The proof of Proposition 3.3 falls into two parts, the first of which is analogous to that of Proposition 2.7, and the second of which corresponds to Proposition 2.9.

Consider colouring the cycle C_{2p+1} with R colours per vertex, but with overlap Λ_j between vertices v_j and v_{j+1} ($j = 0, 1, \ldots, 2p-1$) and overlap Λ_{2p} between vertices v_{2p} and v_0. We call this an $[R, (\Lambda_0, \ldots, \Lambda_{2p})]$ or, for short, an $[R, \Lambda]$ colouring. Denote the minimum of the Λ_j by $\Lambda_{\min}$ and the mean of the Λ_j by $\overline{\Lambda}$. Whereas in the case of constant overlap we showed that $N \geq 2R - \Lambda$, it is clear that, in the case of variable overlap, $N \geq 2R - \Lambda_{\min}$.

Proposition 3.3 $\chi_{R,\Lambda}(C_{2p+1}) \geq \max\left\{2R - \Lambda_{\min}, \left\lceil \frac{(2p+1)(R-\overline{\Lambda})}{p} \right\rceil\right\}.$

Proof Let μ be an $[R, (\Lambda_0, \ldots, \Lambda_{2p})]$ colouring of C_{2p+1} using $N = 2R - \Lambda_{\min} + s$ colours.

For $i = 0, 1, \ldots, 2p$, let S_i be the set of colours at v_i.

Consider $S_0 \cap S_2$; S_0 contains $R - \Lambda_0$ elements disjoint from S_1, and S_2 contains $R - \Lambda_1$ disjoint from S_1, and there are only $R - \Lambda_{\min} + s$ such elements available.

Then $|S_0 \cap S_2| \geq (R - \Lambda_0) + (R - \Lambda_1) - (R - \Lambda_{\min} + s) = R - (\Lambda_0 + \Lambda_1 + s) + \Lambda_{\min}$.

Similarly, for $i = 1, \ldots, p-1$,

$$|S_{2i} \cap S_{2i+2}| \geq R - (\Lambda_{2i} + \Lambda_{2i+1} + s) + \Lambda_{\min}.$$

Thus, arguing as in Chapter 2,

$$|S_0 \cap S_{2p}| \geq R - (\Lambda_0 + \Lambda_1 + \ldots\ldots. + \Lambda_{2p-1} + ps) + p\Lambda_{\min}.$$

But $\quad |S_0 \cap S_{2p}| = \Lambda_{2p}.$

Then $\quad \Lambda_{2p} \geq R - (\Lambda_0 + \Lambda_1 + \ldots + \Lambda_{2p-1} + ps) + p\Lambda_{\min};$

$$ps \geq R - (\Lambda_0 + \ldots + \Lambda_{2p}) + p\Lambda_{\min};$$

$$s \geq \frac{R - (2p+1)\overline{\Lambda}}{p} + \Lambda_{\min}.$$

Since $\quad N = 2R - \Lambda_{\min} + s,$

$$N \geq 2R - \Lambda_{\min} + \frac{R - (2p+1)\overline{\Lambda}}{p} + \Lambda_{\min} = \frac{(2p+1)(R-\overline{\Lambda})}{p},$$

and, since N is an integer,

$$N \geq \left\lceil \frac{(2p+1)(R-\overline{\Lambda})}{p} \right\rceil.$$

Moreover, since $N \geq 2R - \Lambda_{\min}$, we have

$$N \geq \max\left\{2R - \Lambda_{\min}, \left\lceil \frac{(2p+1)(R-\overline{\Lambda})}{p} \right\rceil\right\}. \qquad \blacksquare$$

We now proceed to evaluate $\chi_{r,\lambda}(W_{2p+2})$. The range $\frac{pr}{2p+1} < \lambda < \frac{(p+1)r}{2p+1}$ is the most difficult to evaluate; we deal with the other possible values of λ first.

Proposition 3.4 (Case 1)

$$\chi_{r,\lambda}(W_{2p+2}) = \left\{ \begin{array}{cc} 3r-4\lambda+\left\lceil \frac{r-2\lambda}{p} \right\rceil & \left(0 \le \lambda \le \frac{r}{p+2}\right) \\ 3(r-\lambda) & \left(\frac{r}{p+2} \le \lambda \le \frac{pr}{2p+1}\right) \end{array} \right\} \quad (1)$$

Proof If $\lambda = 0$, then the rim vertices require $\chi_{r,0}(C_{2p+1}) = \left\lceil \frac{(2p+1)r}{p} \right\rceil$ colours, all distinct from the hub colours; thus

$$\chi_{r,0}(W_{2p+2}) = r + \left\lceil \frac{(2p+1)r}{p} \right\rceil = 3r + \left\lceil \frac{r}{p} \right\rceil.$$

Now let $0 < \lambda \le \frac{pr}{2p+1}$. Then $r \ge \left\lceil \frac{(2p+1)\lambda}{p} \right\rceil$, and so, by Theorem 2.1, C_{2p+1} has a $[\lambda, 0]$ colouring using at most r colours; that is, the hub component θ may be chosen to be a

$[\lambda, 0]$

colouring. Then the rim component ω is a $[r-\lambda, \lambda]$ colouring of C_{2p+1}, and requires

$$\max\left\{2(r-\lambda)-\lambda, \left\lceil \frac{(2p+1)((r-\lambda)-\lambda)}{p} \right\rceil\right\} = \max\left\{2r-3\lambda, \left\lceil \frac{(2p+1)(r-2\lambda)}{p} \right\rceil\right\} \quad \text{colours.}$$

Thus, (counting the colours of θ also) we have

$$\chi_{r,\lambda}(W_{2p+2}) \le \max\left\{3(r-\lambda), 3r-4\lambda+\left\lceil \frac{r-2\lambda}{p} \right\rceil\right\}.$$

Moreover, if θ were chosen to have overlaps, then the mean and minimum overlaps of ω would be reduced, and by Proposition 3.3 the number of colours required by ω would be at least as great. Thus

$$\chi_{r,\lambda}(W_{2p+2}) = \max\left\{3(r-\lambda), 3r-4\lambda+\left\lceil \frac{r-2\lambda}{p} \right\rceil\right\}$$

$$= \left\{ \begin{array}{cc} 3r-4\lambda+\left\lceil \frac{r-2\lambda}{p} \right\rceil & \left(0 \le \lambda \le \frac{r}{p+2}\right) \\ 3(r-\lambda) & \left(\frac{r}{p+2} \le \lambda \le \frac{pr}{2p+1}\right) \end{array} \right\} \quad \blacksquare$$

Proposition 3.5 (Case 2)

$$\chi_{r,\lambda}(W_{2p+2}) = 2r-\lambda \qquad \left(\frac{(p+1)r}{2p+1} \le \lambda \le r\right). \quad (2)$$

Proof In this case, we find the number of overlaps required by the hub component; we note that since the rim component has $r-\lambda$ colours per vertex and thus a maximum overlap of $r-\lambda$, the minimum possible overlap for the hub component is $2\lambda - r$. This is possible when the number of colours available for the hub component (namely r) is enough for a $[\lambda, 2\lambda - r]$ colouring; by Theorem 2.1, this requires

$$r \geq \max\left\{2\lambda-(2\lambda-r), \left\lceil \frac{(2p+1)(r-\lambda)}{p} \right\rceil\right\} = \max\left\{r, \left\lceil \frac{(2p+1)(r-\lambda)}{p} \right\rceil\right\},$$

and this is true in the range $\frac{(p+1)r}{2p+1} \leq \lambda \leq r$; this makes the rim component ω an $[r-\lambda, r-\lambda]$ colouring, requiring just $r-\lambda$ colours. Since ω cannot possibly use fewer than $r-\lambda$ colours, we have shown that, for this range of λ,

$$\chi_{r,\lambda}(W_{2p+2}) = r+(r-\lambda) = 2r-\lambda. \qquad \blacksquare$$

In order to deal with the range $\frac{pr}{2p+1} < \lambda < \frac{(p+1)r}{2p+1}$, we require the following lemma, which allows us to deal with all possibilities except those with $r=2$ and with $p=1$; these peculiarities are considered last.

Lemma 3.6 Let $r \geq 3, p \geq 2, L = \left\lfloor \frac{pr}{2p+1} \right\rfloor$ and $\lambda \geq \frac{pr}{2p+1}$; then

(i) $L \geq \frac{r}{2p+1}$, (3)

(ii) $2(r-\lambda)-L \geq \left\lceil \frac{(2p+1)(r-\lambda-L)}{p} \right\rceil$. (4)

Proof (i) Note that

$\frac{pr}{2p+1} - \frac{r}{2p+1} = \frac{(p-1)r}{2p+1} \geq 1$ whenever $p \geq 2$ and $r \geq 3$, except for the cases $(p, r) = (2, 3), (2, 4), (3, 3)$. Thus, (3) holds except possibly for these three instances; but it is straightforward to check that (3) does in fact hold in those instances.

(ii) From (3) we obtain

$$(p+1)L \geq \frac{(p+1)r}{2p+1}.$$

Since $\lambda \geq \frac{pr}{2p+1}$, it follows that $r-\lambda \leq \frac{(p+1)r}{2p+1}$ and hence

$(p+1)L \geq r-\lambda;$

$(2p(r-\lambda)+(p+1)L \geq (2p+1)(r-\lambda);$

$$(2p(r-\lambda)-pL \geq (2p+1)(r-\lambda-L);$$

$$2(r-\lambda)-L \geq \frac{(2p+1)(r-\lambda-L)}{p},$$

and since $2(r-\lambda)-L$ is an integer, (4) follows. ■

Proposition 3.7 (Case 3)

Let $\frac{pr}{2p+1} < \lambda < \frac{(p+1)r}{2p+1}$ and $p \geq 2, r \geq 3$.

Then $\chi_{r,\lambda}(W_{2p+2}) = 3r - 2\lambda - \left\lfloor \frac{pr}{2p+1} \right\rfloor$. (5)

Proof Let the hub component be a λ-colouring of C_{2p+1} using the hub colours 1, ..., r, with overlap Λ. By Proposition 3.3, $r \geq \max\left\{2\lambda - \Lambda_{\min}, \left\lceil \frac{(2p+1)(\lambda-\overline{\Lambda})}{p} \right\rceil\right\}$; thus $\Lambda_{\min} \geq 2\lambda - r$ and $\overline{\Lambda} \geq \lambda - \frac{pr}{2p+1}$. Let $L = \left\lfloor \frac{pr}{2p+1} \right\rfloor$. Now, the rim component ω must have mean overlap at most $\frac{pr}{2p+1}$ and maximum overlap at most $\lambda-(2\lambda-r) = r-\lambda$. Let N be the number of colours required by ω. The first of these inequalities implies that

$$N \geq \max\left\{2(r-\lambda) - \left\lfloor \frac{pr}{2p+1} \right\rfloor, \left\lceil \frac{(2p+1)(r-\lambda-\frac{pr}{2p+1})}{p} \right\rceil\right\}$$

$$= \max\left\{2(r-\lambda)-L, \left\lceil \frac{(2p+1)(r-\lambda)}{p} \right\rceil - r\right\}. \text{ Thus,}$$

$$\chi_{r,\lambda}(W_{2p+2}) \geq r + 2(r-\lambda) - L = 3r - 2\lambda - \left\lfloor \frac{pr}{2p+1} \right\rfloor.$$

But, by Lemma 3.6,

$$2(r-\lambda)-L \geq \left\lceil \frac{(2p+1)(r-\lambda-L)}{p} \right\rceil$$

and so by Theorem 2.1 (since these quantities are integers) there is a constant–overlap $[r-\lambda, L]$ colouring ω using $2(r-\lambda)-L$ colours. The result follows. ■

Finally, we tackle the 'peculiarities'. We note first that if $r \leq 2$ and $\frac{pr}{2p+1} < \lambda < \frac{(p+1)r}{2p+1}$, then the only possibility is $r = 2$, $\lambda = 1$.

Proposition 3.8 (Case 4)

$$\chi_{2,1}(W_{2p+2}) = 5, \text{ for all } p. \tag{6}$$

Proof Let the hub component have the colours 1, 2 at the hub and colour 1 at each rim vertex. Then the rim component can be a [1, 0] colouring of C_{2p+1} using $\chi(C_{2p+1}) = 3$ colours. Hence, $\chi_{2,1}(W_{2p+2}) \leq 5$.

To show the lower bound, note that the above colouring is (apart from a rearrangement of colours) the only [2, 1] colouring of W_{2p+2} where the hub component θ has constant overlap. Thus, if there is a colouring using four colours, then the hub component θ must have variable overlap. Suppose that θ has this property; then we may assume that an odd number of rim vertices take the colour 1 and an even number take the colour 2. Consider the edges of C_{2p+1} in cyclic order; the colour must change an even number of times, hence θ has an odd number of overlaps of size 1 and an even number of overlaps of size 0. However, the same must be true of the rim component ω (since our assumption of four colours in total implies that ω uses just two colours). This would imply that the combined colouring has an even number of overlaps of size 1 on the rim edges, contradicting the requirement that it is a [2, 1] colouring. Thus ω must use more than two colours. ■

Finally, still keeping the assumption $\frac{pr}{2p+1} < \lambda < \frac{(p+1)r}{2p+1}$, we tackle the wheel with $p = 1$ (that is, the graph $W_4 = K_4$) and $r \geq 3$. This is the only case where (in some circumstances) the hub and rim components must necessarily have variable overlap in order to achieve the bound. We establish exactly which these are.

Proposition 3.9 (Case 5) Let $\frac{r}{3} < \lambda < \frac{2r}{3}$, and $r \geq 3$. Then

$$\chi_{r,\lambda}(W_4) = 3r - 2\lambda - \left\lfloor \frac{r}{3} \right\rfloor. \tag{7}$$

Moreover, in order to obtain a colouring with $\chi_{r,\lambda}(W_4)$ colours, it is necessary to use variable–overlap components when $r = 3L + 2$ and $\lambda = L + 1$; for all other values, constant–overlap components are sufficient.

Proof Let θ be a λ–colouring of C_3 using the hub colours 1, ..., r, with overlap Λ. By Proposition 3.3, $r \geq \max\{2\lambda - \Lambda_{\min}, \lceil 3(\lambda - \overline{\Lambda} \rceil\}$ thus $\Lambda_{\min} \geq 2\lambda - r$ and $\overline{\Lambda} \geq \lambda - \frac{r}{3}$.

Let $L = \left\lfloor \frac{r}{3} \right\rfloor$. Arguing as in the proof of Proposition 3.7, the $(r - \lambda)$-colouring ω requires at least $2(r - \lambda) - L$ colours. Thus

$$\chi_{r,\lambda}(W_4) \geq 3r - 2\lambda - L.$$

We establish the circumstances under which variable-overlap components are required. To do this, we let $r = 3L + q$, where $q = 0$, 1 or 2, and $\lambda = L + k$ where $1 \leq k \leq L$ if $q = 2$, otherwise $1 \leq k \leq L - 1$.

<u>Subcase 5a</u> $r = 3L$.

Here, the hub component has mean overlap at least $\lambda - \frac{r}{3} = k$. Thus the rim component ω has mean overlap at most $\lambda - k = L$. Then we must choose ω to have constant overlap L.

Then
$$N(\omega) \geq \{\max 2(r - \lambda) - L, 3(r - \lambda - L)\}$$
$$= \max\{2(r - \lambda) - L, 2(r - \lambda) + r - \lambda - 3L\}$$
$$= \max\{2(r - \lambda) - L, 2(r - \lambda) - L - k\} = 2(r - \lambda) - L$$

Thus, $\chi_{r,\lambda}(W_4) = 3r - 2\lambda - L$, and variable-overlap components are not required.

<u>Subcase 5b</u> $r = 3L + 1$.

The hub component here has mean overlap at least $\lambda - \frac{r}{3} = k - \frac{1}{3}$, so ω has mean overlap at most $L + \frac{1}{3}$. If we can choose exactly this mean overlap (so that ω has two overlaps of size L and one of size $L + 1$), then

$$N(\omega) \geq \max\{2(r - \lambda) - L, 2(r - \lambda) + r\_\lambda - 3L - 1\}$$
$$= \max\{(2r - \lambda) - L, 2(r - \lambda) - L - k\} = (2r - \lambda) - L$$

Alternatively, if we choose the hub and rim components to have constant overlaps k and L respectively, then denote this rim component by v. Then

$$N(v) = \max\{2(r - \lambda) - L, 3(r - \lambda - L)\}$$

$$= \max\{2(r-\lambda)-L, 2(r-\lambda)+r-\lambda-3L\}$$

$$= \max\{2(r-\lambda)-L, 2(r-\lambda)-L-k+1\}$$

$$= 2(r-\lambda)-L, \text{ as before.}$$

Thus, $\chi_{r,\lambda}(W_4) = 3r-2\lambda-L$, and again variable-overlap components are not required.

Subcase 5c $\quad r = 3L+2$.

The hub component here has mean overlap at least $\lambda - \frac{r}{3} = k - \frac{2}{3}$, so ω has mean overlap at most $L + \frac{2}{3}$. If we can choose exactly this mean overlap (so that now ω has two overlaps of size $L + 1$ and one of size L), then

$$N(\omega) \geq \max\{2(r-\lambda)-L, 2(r-\lambda)+r-\lambda-3L-2\}$$

$$= \max\{2(r-\lambda)-L, 2(r-\lambda)-L-k\} = 2(r-\lambda)-L.$$

Again, if we choose the hub and rim components to have constant overlaps k and L respectively, then denote this rim component by v. We obain

$$N(v) = \max\{2(r-\lambda)-L, 3(r-\lambda-L\}$$

$$= \max\{2(r-\lambda)-L, 2(r-\lambda)+r-\lambda-3L\}$$

$$= \max\{2(r-\lambda)-L, 2(r-\lambda)-L-k+2\}.$$

If $k \geq 2$, then as before, this choice requires the same number of colours as the variable-overlap component ω.

However, if $k = 1$, then the constant-overlap choice forces one extra colour. Thus, in this case, we need to check that the variable-overlap colourings do exist.

Note that the hub component is required to have two overlaps of size 0 and one of size 1. Since $r \geq 3$, part (i) of Proposition 2.6 applies; hence by the observation, the hub component can be chosen to have these overlap sizes, using $3\lambda - 1 = r$ colours. Then the rim component requires two overlaps of size $L + 1$ and one of size L. Since $L+1 > \frac{r-\lambda}{2} = L + \frac{1}{2}$, part (ii) of Proposition 2.6 applies, hence by the observation , the hub component can be chosen to have these overlap sizes, using

$2(r-\lambda)-(L+1)+1=2(r-\lambda)-\left\lfloor \frac{r}{3} \right\rfloor$ colours. This completes the proof. ■

Theorem 3.10 The value of $\chi_{r,\lambda}(W_{2p+2})$ requires five expressions, depending on the values of λ and r.

(i) For $r=2$ and $\lambda=1$, we have $\chi_{2,1}(W_{2p+2})=5$. In all other cases:

(ii) If $0 \le \lambda \le \frac{r}{p+2}$, then $\chi_{r,\lambda}(W_{2p+2})=3r-4\lambda+\left\lceil \frac{r-2\lambda}{p} \right\rceil$;

(iii) if $\frac{r}{p+2} \le \lambda \le \frac{pr}{2p+1}$, then $\chi_{r,\lambda}(W_{2p+2})=3(r-\lambda)$;

(iv) if $\frac{pr}{2p+1} \le \lambda \le \frac{(p+1)r}{2p+1}$, then $\chi_{r,\lambda}(W_{2p+2})=3r-2\lambda-\left\lfloor \frac{pr}{2p+1} \right\rfloor$;

(v) if $\lambda \ge \frac{(p+1)r}{2p+1}$, then $\chi_{r,\lambda}(W_{2p+2})=2r-\lambda$. ■

(Note that if $p=1$, Case (iii) does not occur.)

Annex 3.1 gives, for some values of $[r, \lambda]$, the minimum number of colours required for W_6, W_8, W_{10} and W_{12}.

Annex 3.1

Wheel Colourings

$[r,\lambda]$	W_6 N	W_8 N	W_{10} N	W_{12} N
[1, 0]	4	4	4	4
[2, 0]	7	7	7	7
[2, 1]	4	4	4	4
[3, 0]	11	10	10	10
[3, 1]	6	6	6	6
[3, 2]	4	4	4	4
[4, 0]	14	14	13	13
[4, 1]	9	9	9	9
[4, 2]	7	7	7	7
[4, 3]	5	5	5	5
[5, 0]	18	17	17	16
[5, 1]	13	12	12	12
[5, 2]	9	9	9	9
[5, 3]	7	7	7	7
[5, 4]	6	6	6	6
[6, 0]	21	20	20	20
[6, 1]	16	16	15	15
[6, 2]	12	12	12	12
[6, 3]	10	10	10	10
[6, 4]	8	8	8	8
[6, 5]	7	7	7	7
[7, 0]	25	24	23	23
[7, 1]	20	19	19	18
[7, 2]	15	15	15	15
[7, 3]	13	12	12	12
[7, 4]	11	10	10	10
[7, 5]	9	9	9	9
[7, 6]	8	8	8	8
[8, 0]	28	27	26	26
[8, 1]	23	22	22	22
[8, 2]	18	18	18	18
[8, 3]	15	15	15	15
[8, 4]	13	13	13	13
[8, 5]	11	11	11	11
[8, 6]	10	10	10	10
[8, 7]	9	9	9	9
[9, 0]	32	30	30	29
[9, 1]	27	26	25	25
[9, 2]	22	21	21	21
[9, 3]	18	18	18	18
[9, 4]	16	16	15	15
[9, 5]	14	14	13	13
[9, 6]	12	12	12	12
[9, 7]	11	11	11	11
[9, 8]	10	10	10	10

Chapter 4: Overlap Colourings and Homomorphisms

In this chapter we investigate the place of overlap chromatic numbers in the classification of graphs.

Let $r \geq 1,\ 0 \leq \lambda \leq r$. We have seen (Proposition 1.2) that if there is a homomorphism from a graph G to a graph H, then $\chi_{r,\lambda}(G) \leq \chi_{r,\lambda}(H)$. Moreover, if also H is isomorphic to a subgraph of G, then $\chi_{r,\lambda}(G) \geq \chi_{r,\lambda}(H)$, and so $\chi_{r,\lambda}(G) = \chi_{r,\lambda}(H)$. The *core* of a graph G is the smallest subgraph C of G such that there is a homomorphism from G to C; this is unique up to isomorphism (see [9]), and by the above remarks has the same overlap chromatic numbers as G. Thus, classifying graphs by their cores is certainly at least as fine a classification as by their overlap chromatic numbers. We shall now show that graphs with different cores can have the same overlap chromatic numbers (so that classifying by the former is strictly finer than by the latter). We say that G and H have the same *multichromatic profile* if $\chi_{r,0}(G) = \chi_{r,0}(H)\ \ (r \geq 1)$ and the same *overlap profile* if $\chi_{r,\lambda}(G) = \chi_{r,\lambda}(H)\ \ (r \geq 1,\ 0 \leq \lambda \leq r)$.

Theorem 4.1 Let G be a graph with the same multichromatic profile as C_{2p+1}, and containing C_{2p+1} as a subgraph. Then G also has the same overlap profile as C_{2p+1}.

Proof. Let $r \geq 1,\ 0 \leq \lambda \leq r$. Since C_{2p+1} is a subgraph of G, $\chi_{r,\lambda}(G) \geq \chi_{r,\lambda}(C_{2p+1})$; we shall now construct an $[r, \lambda]$-colouring of G using $\chi_{r,\lambda}(C_{2p+1})$ colours.

Let n be such that $np + i = r - \lambda$ where $1 \leq i \leq p$; then $Q_{\lambda,n}(i)$ (see page 25) is an efficient $[r, \lambda]$-colouring of C_{2p+1}. By the proof of Theorem 2.10, if $\lambda > n$, then $Q_{\lambda,n}(i) = \pi[i+1, 1] + (\lambda - n - 1)\pi[1, 1] + n\pi[p+1, 1]$. Since G has the same multichromatic profile as C_{2p+1}, G must have an $[i, 0]$ colouring $\theta[i, 0]$ using $2i + 1$ colours and a $[p, 0]$ colouring $\theta[p, 0]$ using $2p + 1$ colours. The complement φ of $\theta[i, 0]$ is an $[i + 1, 1]$ colouring of G using $2i + 1$ colours, and the complement ψ of $\theta[p, 0]$ is a $[p + 1, 1]$ colouring of G using $2p + 1$ colours. The colouring $\pi[1, 1]$ is available for any graph, using one colour. Thus, the colouring

$$\varphi + (\lambda - n - 1)\pi[1,1] + n\psi$$

of G uses the same palette as $Q_{\lambda,n}(i)$, and thus $\chi_{r,\lambda}(G) \leq \chi_{r,\lambda}(C_{2p+1})$ as required.

If $n \geq \lambda$, then $Q_{\lambda,n}(i) = \pi[i,0] + (n-\lambda)\pi[p,0] + \lambda[p+1,1]$, and a similar argument again shows that $\chi_{r,\lambda}(G) \leq \chi_{r,\lambda}(C_{2p+1})$. Thus G has the same overlap profile as C_{2p+1}. ■

Corollary 4.2 Let G be a graph with $\chi_p(G) = 2p+1$, and containing C_{2p+1} as a subgraph. Then G has the same overlap profile as C_{2p+1}.

Proof Theorem 2 of [23] states that if G has an edge, then $\chi_n(G) \geq \chi_{n-1}(G)$ for all $n > 1$. Now since G contains an odd cycle, $\chi_1(G) \geq 3$. Since also $\chi_p(G) = 2p+1$, it follows that $\chi_q(G) = 2q+1$ $(1 \leq q \leq p)$. Hence G has the same multichromatic profile as C_{2p+1}, and the result follows from Theorem 4.1. ■

Bangles

We now consider a class of graphs which we term *bangles*. These are a sub-class of the class of *series-parallel graphs*, whose chromatic properties (particularly their circular chromatic numbers) have been studied (see, for example, Pan and Zhu [19]).

Informally, a bangle is an odd cycle of odd cycles, 'welded' together at points as far apart as possible on each cycle. More formally, the bangle $B(2q+1, 2p+1)$ is formed from $2q+1$ copies of the cycle C_{2p+1} as follows. The copies of the cycle are denoted by $C_{12}, C_{23}, \ldots, C_{2q+1,1}$; a vertex of $C_{2q+1,1}$ is identified with a vertex of C_{12} to form a 'weld vertex' W_1; a vertex of C_{12} at distance p from W_1 on C_{12} is similarly identified with a vertex of C_{23} to form a weld vertex W_2; and the welding process continues cyclically, so that W_{2q+1} welds $C_{2q,2q+1}$ with $C_{2q+1,1}$. (We say that two vertices are at distance p in a graph if the shortest path connecting them has p edges.) Thus, for example, the bangle $B(3, 7)$ may be drawn as follows.

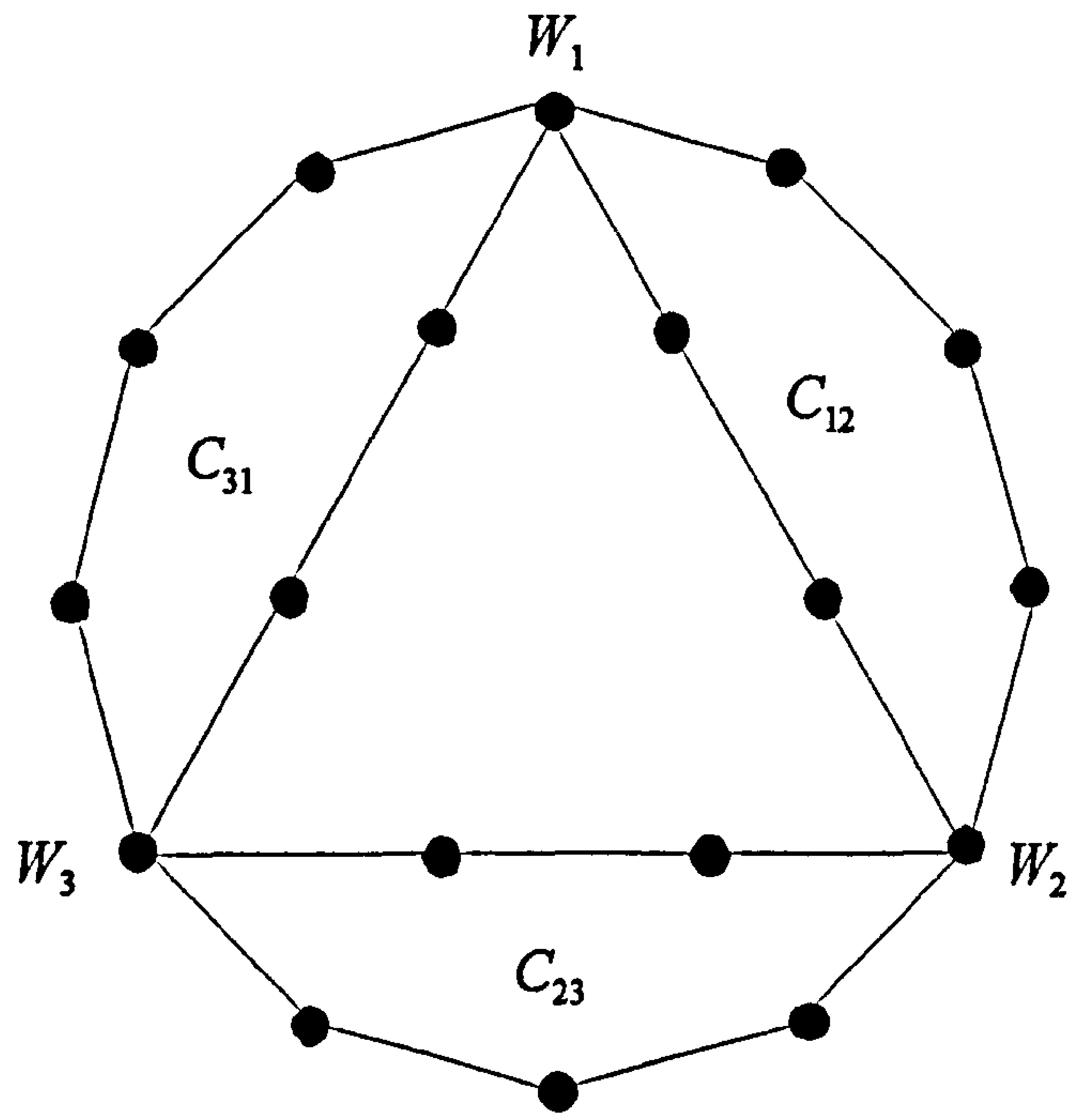


Figure 1

Proposition 4.3 Let $p \geq 2$. Then there is no homomorphism from $B(3, 2p + 1)$ to C_{2p+1}.

Proof Let us allocate colour i to vertex v_i of C_{2p+1} $(0 \leq i \leq 2p)$. Then, a homomorphism from $B(3, 2p + 1)$ to C_{2p+1} may be regarded as a $[1, 0]$ colouring of $B(3, 2p + 1)$ using palette $\{0, 1, ..., 2p\}$ such that the colours occur cyclically, in clockwise or anticlockwise order, on each of C_{12}, C_{23} and C_{31}. We may assume without loss of generality that the weld W_1 is coloured 0 and the cycle C_{12} is coloured clockwise; thus W_2 is coloured $p + 1$. Now if C_{23} is coloured clockwise, then W_3 takes the colour 1, while if C_{23} is coloured anticlockwise, then W_3 takes the colour 0. In neither case can the colouring of C_{31} be completed cyclically.

Thus, no homomorphism is possible. ■

Thus, the core of $B(3, 2p + 1)$ is *not* C_{2p+1}. Nevertheless, we shall now show the following.

Theorem 4.4 For any $q \geq 1$, $p \geq 2$, the bangle $B(2q + 1, 2p + 1)$ has the same overlap profile as the cycle C_{2p+1}.

Before we prove this theorem, we require the following lemma.

Lemma 4.5 Let μ be the cyclic $[p, 0]$ colouring of C_{2p+1} and let vertices v, w be at distance p.

(i) If $p = 2a - 1$, then $|\mu(v) \cap \mu(w)| = a - 1$;

(ii) If $p = 2a$, then $|\mu(v) \cap \mu(w)| = a$.

Proof We may assume without loss of generality that $v = v_0$ and $w = v_p$, since the colour sets at any other pair of vertices separated by p edges are related to those at v_0 and v_p by adding some constant modulo $(2p + 1)$. Thus we may assume

$$\mu(v_0) = \{1, 2, ..., p\}; \qquad \mu(v_1) = \{p+1, p+2, ..., 2p\},$$

$$\mu(v_2\} = \{2p+1, 2p+2, ..., p-1\}; \quad \mu(v_3) = \{p, p+1, ..., 2p-1\},$$

and so on, so that

$$\mu(v_{2j}) = \{2p+2-j, ..., 2p+1, 1, ..., p-j\} \ (1 \le j < p),$$

$$\mu(v_{2j+1}) = \{p+1-j, ..., 2p-j\} \ (1 \le j < p).$$

Thus, if $p = 2a - 1$, then $\mu(v_0) = \{1, 2, ..., 2a-1\}$ and

$$\mu(v_p) = \mu(v_{2a-1}) = \{p-a+2, ..., p, ..., 2p-a+1\} = \{a+1, ..., 3a-1\},$$

giving $\qquad |\mu(v) \cap \mu(w)| = a - 1;$

if $p = 2a$, then $\qquad \mu(v_0) = \{1, 2, ..., 2a\}$ and

$$\mu(v_p) = \mu(v_{2a}) = \{3a+2, ..., 4a+1, 1, ..., a\},$$

giving $\qquad |\mu(v) \cap \mu(w)| = a.$ ■

Proof of Theorem 4.4

Note first that we need only show the result for $q = 1$, since if $B(3, 2p + 1)$ has an $[r, \lambda]$-overlap colouring using $\chi_{r,\lambda}(C_{2p+1})$ colours, then we may colour the first three weld vertices W_1, W_2 and W_3 of $B(2q + 1, 2p + 1)$ as for $B(3, 2p + 1)$ and then colour subsequent weld vertices by alternating between the colourings of W_2 and W_3 (that is, there is a homomorphism from $B(2q + 1, 2p + 1)$ to $B(3, 2p + 1)$).

Denote $B(3, 2p + 1)$ just by B. We shall now construct a $[p, 0]$ colouring ϕ of B using

$2p + 1$ colours, thus showing that $\chi_{p,0}(B) = 2p+1$. The result then follows from Corollary 4.2. ■

To do this, we need to define a rotation sense in each cycle. As in Figure 2, below, we shall consider the longer paths between the weld points to be drawn round the outside of a drawing of B, and the cycles to be as in Figure 2, with W_1 at the top. We colour the vertices of C_{12} clockwise starting from W_1, so that $\phi(W_1) = \{1,\ldots,p\}$, the next vertex has the colour set $\{p+1,\ldots,2p\}$, etc. There are two cases to consider, depending on the size of the cycle modulo 4. Thus, we now express B as $B(3, 4a-1)$ or as $B(3, 4a+1)$ $(a \geq 1)$.

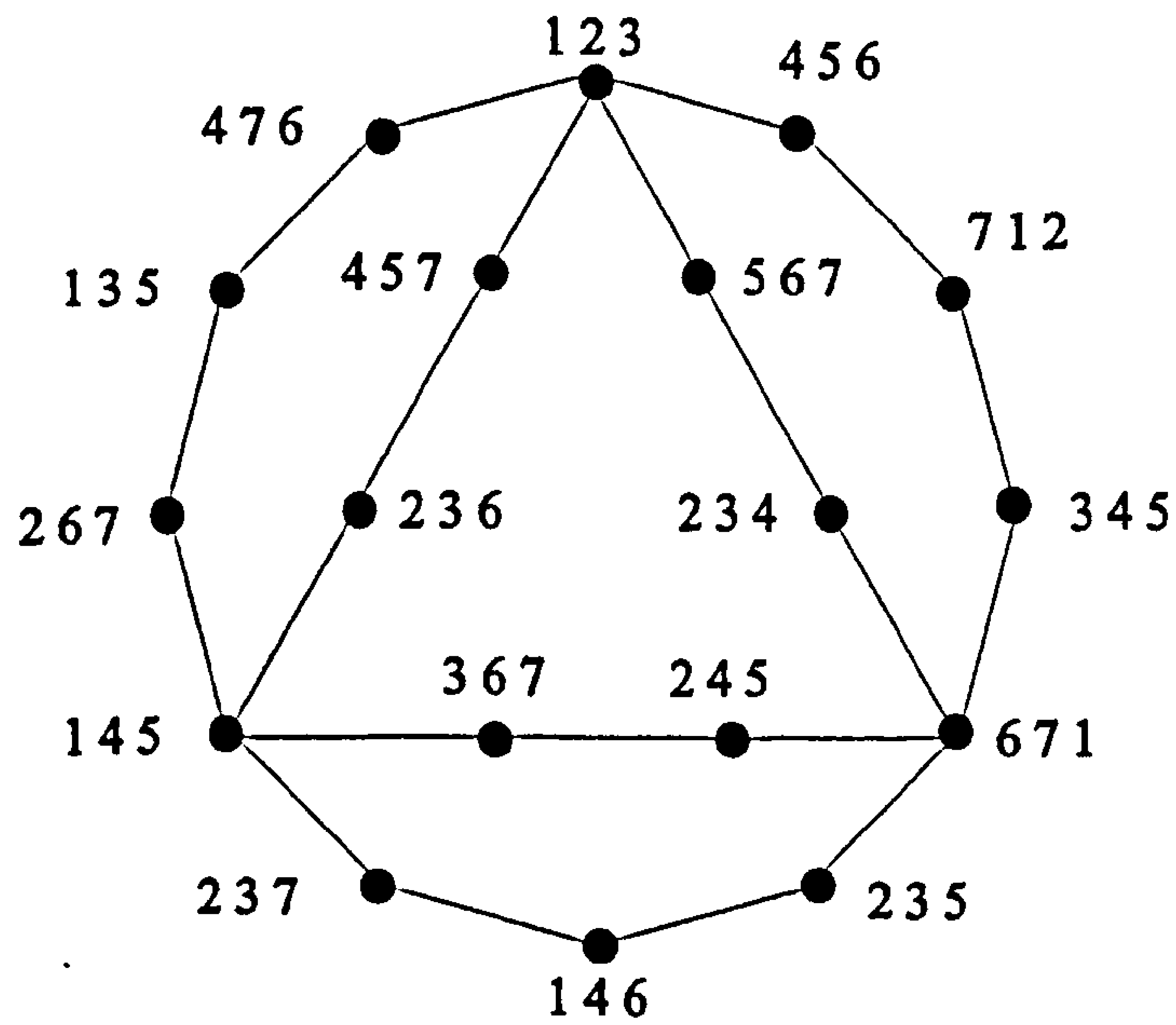


Figure 2: $B(3, 7)[3, 0]$

We take $B(3, 7)[3, 0]$ (Figure 2) as a detailed example. We colour C_{12} with the standard cyclic colouring, proceeding clockwise; the colours of W_1 are 123, and those of W_2 671. Since W_1 and W_2 overlap by one colour, 1, the colours of W_3 must be 145 so as to overlap each of W_1 and W_2 by exactly one colour.

To find suitable colours for C_{31}, we would like to reflect in the axis of symmetry through W_1 and allocate to each vertex of C_{31} the colours of the corresponding vertex of C_{12}. However, this would give the wrong colour set to W_3. Thus, after the reflection, we need to permute the colours so that $\{6, 7, 1\}$ becomes $\{1, 4, 5\}$. One way to do this is to apply the permutation

$$\pi_1 = (23)(46)(57);$$

although not the simplest possibility, it is consistent with a systematic approach which we shall describe later.

We then apply a similar process to the vertices of C_{23}, relecting in the axis of symmetry and applying a suitable permutation to bring $\{1, 2, 3\}$ to $\{1, 4, 5\}$, in this case

$$\pi_2 = (24)(35)(67).$$

A similar argument leads to the colouring of $B(3, 9)[4, 0]$ (Figure 3):

$$\pi_1 = (13)(24)(56)(79) \qquad \pi_2 = (12)(35)(46)(79)$$

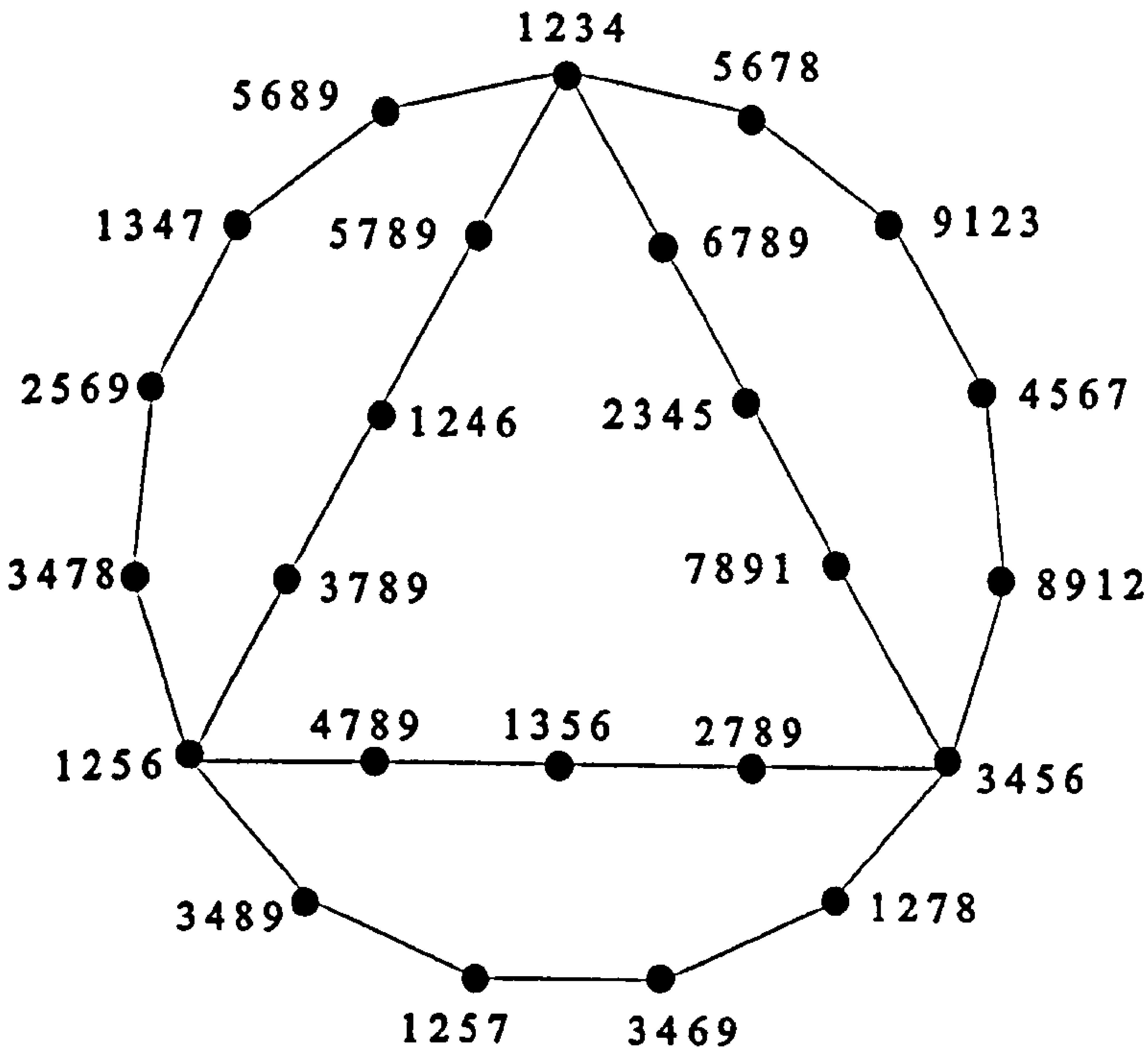


Figure 3: $B(3, 9)[4, 0]$

We form the complements of the above:

The complement of $B(3, 7)[3, 0]$ is $B(3, 7)[4, 1]$, and also has the permutations:

$$\pi_1 = (23)(46)(57), \qquad \pi_2 = (24)(35)(67).$$

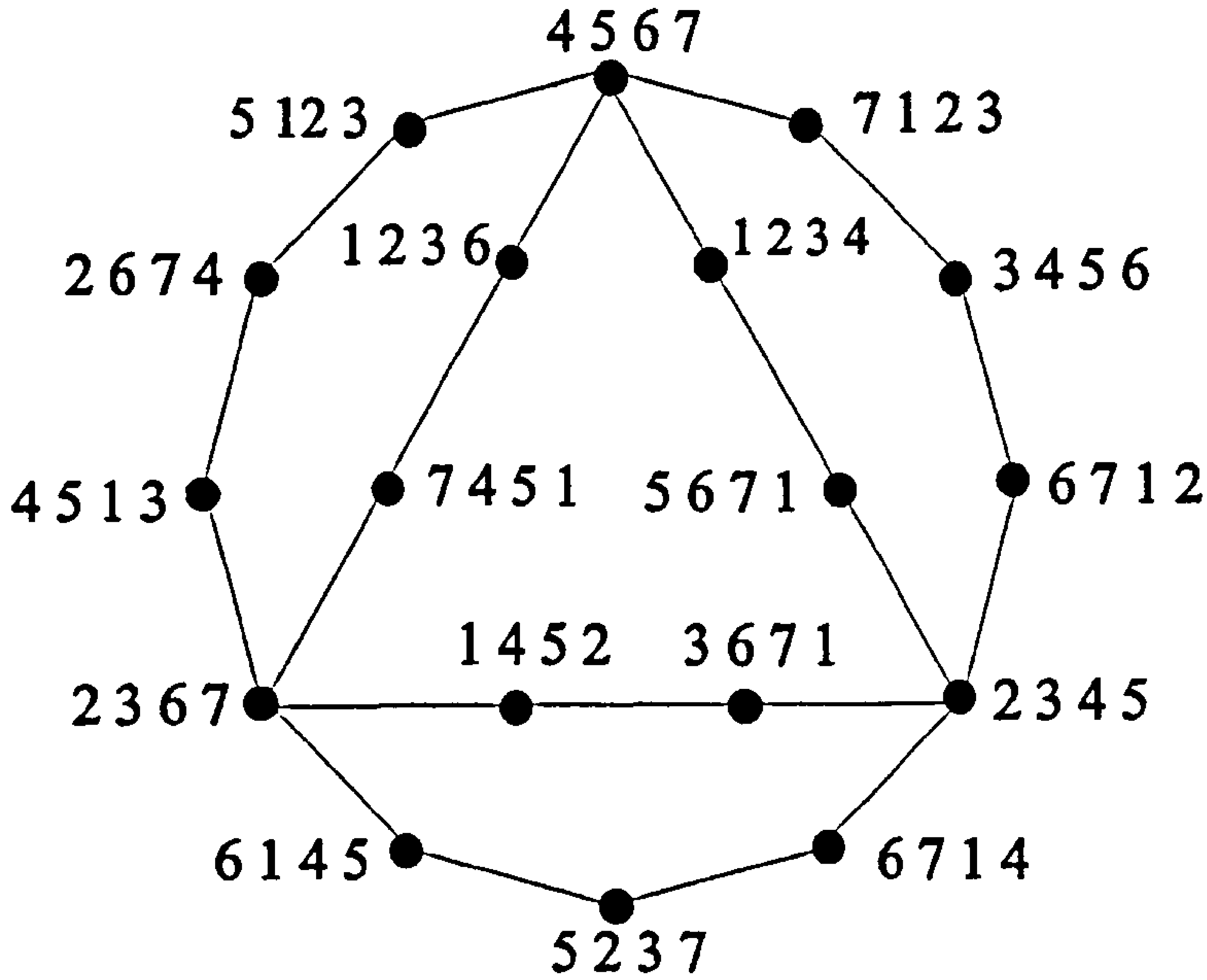


Figure 4: $B(3, 7)[4, 1)$

The complement of $B(3, 9)[4, 0]$ is $B(3, 9)[5, 1]$, which also has the permutations:

$$\pi_1 = (13)(24)(56)(79) \qquad \pi_2 = (12)(35)(46)(79)$$

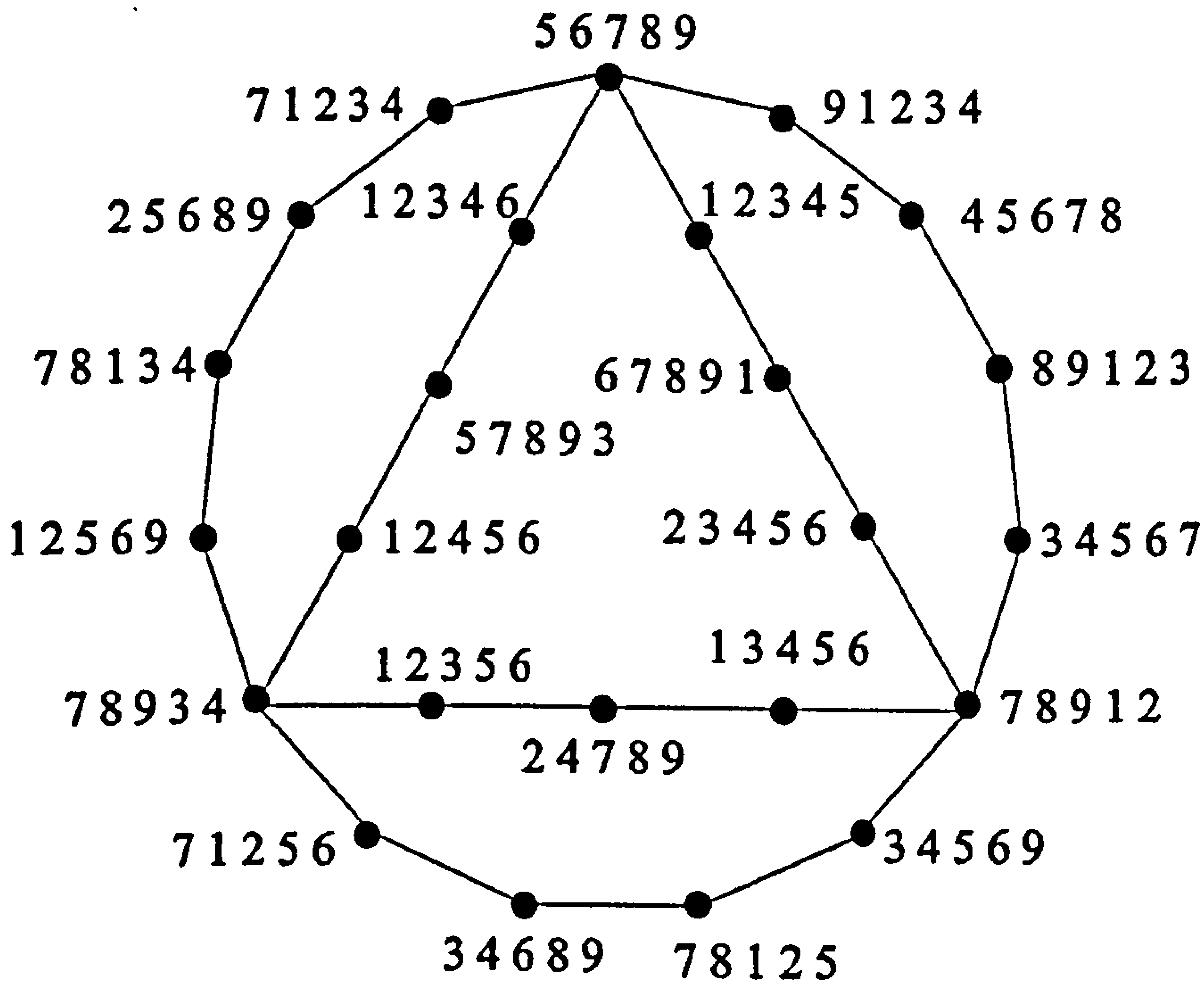


Figure 5: $B(3, 9)[5, 1]$

These examples are typical; we generalize as follows.

$B(3, 7)[3, 0]$ is an example of the general $B(3, 4a-1)[2a-1, 0]$. We generalize the colouring of the latter as follows, beginning with C_{12}:

v_0 (i e W_1)	$1, 2, ..., a-1, a, 2a-1$
v_1	$2a, 2a+1, ..., 4a-2$
v_2	$4a-1, 1, ..., 2a-2$
v_4	$4a-2, ..., 2a-3$
....	
v_{2a} (i e W_2)	$3a, ..., 4a-1,\ 1, ..., a-1.$

The overlap between W_1 and W_2 is thus $1, 2, ..., a-1$. We then colour W_3

$1, 2, ..., a-1, 2a, ..., 3a-1$. This overlaps by the correct number of colours with W_1 and W_2, and one possible choice of permutations is

$$\pi_1 = (1...a-1)(a...2a-1)(2a\ 3a)(2a+1\ 3a+1)...(3a-1\ 4a-1)$$
$$\pi_2 = (1...a-1)(3a...4a-1)(a\ 2a)(a+1\ 2a+1)...(2a-1\ 3a-1)$$

W_1, W_2 and W_3 are all coloured using $4a$ - 1 colours; all colours are used.

$B(3, 9)[4, 0]$ is an example of the general $B(3, 4a+1)[2a, 0]$. We colour the latter:

v_0 (W_1)	$1, 2, ..., 2a$
v_1	$2a+1, ..., 4a$
v_2	$4a+1, 1, ..., 2a-1$
v_3	$2a, ..., 4a-1$
v_5	$2a-1, ..., 4a-2$
....	
v_{2a+1} (W_2)	$a+1, ..., 2a, 2a+1, ..., 3a$

We set W_3 $\quad 1, ..., a, 2a+1, ..., 3a$. This gives a possible set of permutations:

$$\pi_1 = (1\ a+1)(2\ a+2)......(a\ 2a)(2a+1\ 3a)(3a+1\ 4a+1)$$
$$\pi_2 = (a+1\ 2a+1)(a+2\ 2a+2)...(2a\ 3a)(1\ a)(3a+1\ 4a+1)$$

W_1, W_2 and W_3 are all coloured from $4a+1$ colours, without using colours

$3a+1, ..., 4a+1$. ■

Chapter 5: 'Petersen' Graphs $P(p, q)$

We follow Watkins [27] in using the term 'generalized Petersen graph' (sometimes abbreviated to 'GenPet') to refer to graphs (denoted by $P(p, q)$) which may be drawn with an outer cycle of p vertices v_i ($i = 0, \dots, p-1$), each joined to an inner vertex w_i, w_i being joined by an edge to $w_{i+q(\mathrm{mod}\, p)}$. We use the term 'Petersen graphs' loosely to include generalized Petersen graphs in which $p > 5$.

Following [27], we need consider only graphs in which $1 < q < \frac{p}{2}$, since $P(p, p-q)$ is isomorphic to $P(p, q)$. If p is even, $q = \frac{p}{2}$ produces a degenerate graph in which each inner vertex is joined to its opposite, and we do not consider these.

As we shall show, there is a rather small class of GenPets whose overlap parameters can be fully described using Corollary 4.2 (this class includes the Petersen graph itself). In the main, however, the overlap parameters of these graphs seem to be difficult to find. We do, however, describe a systematic approach to the *fractional* chromatic numbers of GenPets; in particular, Theorem 5.8 expresses $\chi_f(P(p, q))$ in terms of the sizes of certain maximal independent vertex sets of $P(p, q)$.

The proof of this result involves showing that a colouring of $P(p, q)$ with $\chi_f(P(p, q))$ colours can always be constructed either as an equable colouring or as a juxtaposition of two equable colourings.

We begin the chapter with an extended description of equable colourings of GenPets, by devices that we call *p-plets*.

5.1 Constructing equable colourings of GenPets

We can sometimes find an equable colouring of $P(p, q)$ where there is no equable colouring of the outer cycle, that is, where there is no equable colouring of C_p. A simple example of this is provided by $P(5, 2)[5, 2]$. There is no equable C_5 [5, 2] colouring, by Corollary 1.5 of Chapter 1, so that we cannot colour the outer vertices with a cycle of single frequency. We can, however, colour them using $C_5[3, 2]$ and $C_5[2, 0]$, whose frequencies are 3 and 2 respectively. We then colour the inner vertices with the same colours but with the frequencies reversed, producing a colouring in which $N = 10$ and $f = 5$ (we note that an equable colouring is not necessarily efficient; the graph in question can be coloured more efficiently but not equably using only 8 colours).

Colours

	Vertices	1	2	3	4	5	6	7	8	9	10
O	v_0	x	x	x			x	x			
U	v_1		x	x	x				x	x	
T	v_2			x	x	x	x				x
E	v_3	x			x	x		x	x		
R	v_4	x	x			x				x	x
I	w_0				x	x	x	x	x		
N	w_1	x				x			x	x	x
N	w_2	x	x				x	x			x
E	w_3		x	x				x	x	x	
R	w_4			x	x		x			x	x

Essentially, this is a juxtaposition of two cyclic colourings $\mu_1 + \mu_2$, where μ_1 cycles through the colours 1, 2, 3, 4, 5, with $d = 1$, and μ_2 cycles through the colours 6, 7, 8, 9, 10, with $d = 2$. (However, μ_1 and μ_2 are not true overlap colourings as they do not give constant numbers of colours to the vertices.) Note that, on renumbering the second colour set

$$6 \mapsto 6,\ 8 \mapsto 7,\ 10 \mapsto 8,\ 7 \mapsto 9,\ 9 \mapsto 10,$$

we produce an isomorphic colouring in which both cycles have $d = 1$.

Colours

	Vertices	1	2	3	4	5	6	7	8	9	10
O	v_0	x	x	x			x			x	
U	v_1		x	x	x			x			x
T	v_2			x	x	x	x		x		
E	v_3	x			x	x		x		x	
R	v_4	x	x			x			x		x
I	w_0				x	x	x	x		x	
N	w_1	x				x		x	x		x
N	w_2	x	x				x		x	x	
E	w_3		x	x				x		x	x
R	w_4			x	x		x		x		x

It might appear that an equable colouring of $(P(p, q))$ necessarily requires more colours than an equable colouring of C_p with the same parameters r, λ, but this is not always so. A counter-example is $P(5, 2)[3, 0]$, which has an equable colouring with $N = 10$, whereas an equable colouring of C_5 requires $N = 15$.

Attempts at finding colourings can be made using actual drawings of graphs, but these occupy much space. We assign colours more compactly by representing vertex-pairs (VP) of outer and inner vertices with sets of p symbols ('p-plets'), each p-plet having the same number of colours assigned, as follows. In a given p-plet, let colour sets S_0, T_0 be assigned to vertices v_0, w_0, respectively. Then we assign the colour set $S_0 + j$ (modulo p) to vertex v_j and colour set $T_0 + j$ (modulo p) to vertex w_j, for $j = 1, \ldots, p-1$. This ensures that each vertex receives the same number of colours, and that no vertex receives the same colour twice. We assign colours using symbols with the following meanings:

- ● colour assigned to outer vertex only;
- ■ colour assigned to inner vertex only;
- ◆ colour assigned to both outer and inner vertices;
- □ colour not assigned to either type of vertex.

The second colouring on the previous page is then expressed ●●●■■ ◆■□◆□.

The cyclic structure of a p-plet means that each p-plet has constant frequency (counting 1 for each ● and ■ and two for each ◆), and so is equable. The number of p-plets will depend on r and λ. In total, we need λ occurrences of ◆. The outer vertex will then require $r - \lambda$ occurrences of ● and the inner vertex $r - \lambda$ occurrences of ■. The number of p-plets must be great enough to accommodate |◆| + |●| + |■| symbols. In any one p-plet, the number of overlaps need not be constant, nor outer and inner overlaps be equal, but the total number of each over all p-plets must be λ.

A simple example makes the method plain. Consider colouring $P(5, 2)$ [6, 1]. One colour in each VP, the overlap colour, is provided by the single ◆; in addition, each of the pair will have a further five colours, so that we have eleven symbols to include. Then we need at least three 5-plets. With the ◆ counting as two, we need to place four colours in each of the three 5-plets, with exactly one overlap among the ● and one overlap among the ■. One example is the following:

◆□●■□ ●●■■□ ■●□●■

In this example, the outer and inner overlaps occur in the second and first 5-plets respectively; there is, then, no further overlap in the third.

	Vertices \ Colours	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
O	v_0	x		x			x	x					x		x	
U	v_1		x		x			x	x					x		x
T	v_2			x		x			x	x		x			x	
E	v_3	x			x					x	x		x			x
R	v_4		x			x	x				x	x		x		
I	w_0	x			x				x	x		x				x
N	w_1		x			x				x	x	x	x			
N	w_2	x		x			x				x		x	x		
E	w_3		x		x		x	x						x	x	
R	w_4			x		x		x	x						x	x

This colouring is a juxtaposition $\mu_1+\mu_2+\mu_3$, each μ_i being described by a single 5-plet representing a cyclic colouring arrangement. (In this case, each 5-plet allocates two colours per vertex, but μ_1 and μ_2 separately do not have the property of constant overlap.) A colouring that can be constructed in the above way is a *p-plet colouring*. Such colourings may be characterized as follows. A *colour class* is the set of vertices that receive a particular colour; then, in a p-plet colouring, each rotational image of any colour class is a colour class. The use of the symbols listed above can greatly reduce the effort required to establish that a given $P(p, q)$ $[r, \lambda]$ has no equable p-plet colouring. As an example, consider $P(5, 2)$ [9, 7]; this is large enough to require considerable trial-and-error on an actual diagram.

We need 7 ◆ + 2 ● + 2 ■. Since there are 11 symbols, we need at least 3 5-plets, and in order to distribute 18 colours (recalling that 7 ◆ counts as 14 colours) equably we need 3, 6, 9 or 18 5-plets. We can eliminate at once:

18 with 1 colour each (because of the ◆);

9 with 2 colours each (too few overlaps between outer vertices and between inners);

6 with 3 colours each (at least one VP will have at least 2 ◆).

This leaves only 3 with 6 each. Only two arrangements are possible. One will have 3 ◆, one 2 ◆ + 2 ●, and the third 2 ◆ + 2 ■. The other arrangement has one 6-plet with 3 ◆, and the other two each 2 ◆ + 1 ● + 1 ■. In either case, the maximum number of possible outer overlaps is then 6, as is the maximum number of inner overlaps, showing that there is no equable *p*-plet colouring.

Some graphs which have a *p*-plet colouring may have a more economical non-*p*-plet colouring; a small example is $P(6, 2)$ [3, 2]. The *p*-plet method gives a colouring using 12 colours. Trial-and-error on the diagram produces a colouring with only 4 colours. It is worth, for the record, listing the colouring:

	Vertices	Colours 1	2	3	4
O	v_0	x	x		x
U	v_1		x	x	x
T	v_2	x		x	x
E	v_3	x	x		x
R	v_4		x	x	x
	v_5	x	x	x	
I	w_0	x	x	x	
N	w_1	x	x		x
N	w_2		x	x	x
E	w_3	x	x	x	
R	w_4	x		x	x
	w_5	x		x	x

Note that, though equable, this colouring does not possess the cyclicity property of *p*-plet colourings.

New Colourings from Old

Complements

The colouring obtained by complementing a p-plet colouring is again a p-plet colouring, obtained by exchanging ● with ■ and ◆ with □. As in Chapter 1, we have the relations:

$$N_1 = N; \qquad n_1 = n; \qquad r_1 = N - r; \qquad \lambda_1 = N - 2r + \lambda,$$

which hold provided that there is no colour that appears at every vertex.

For example, from the colouring of $P(5, 2)$ with $[r, \lambda] = [4, 1]$, in which $N = 10$, we can obtain the colouring with $[r, \lambda] = [6, 3]$.

Chaining

We may use the homomorphism construction referred to as 'wrapping' or 'chaining' in Chapter 1 to obtain a colouring of $P(ap, q)$ from a copies of a colouring of $P(p, q)$. Annex 1.1 illustrates the construction of the graph $P(14, 3)$ from two copies of $P(7, 3)$, and it is straightforward to check that if each copy is given an $[r, \lambda]$ colouring, then the inserted edges (coloured green in the figure) have overlap λ, so that we obtain an $[r, \lambda]$ colouring of $P(14, 3)$.

GenPet Automorphisms

The graph $P(p, q)$ has p-fold rotation symmetry; however, the colourings of $P(p, q)$ that arise from a given colouring μ differ only trivially from μ itself.

There are, however, bijections of some GenPets that produce non-trivial new drawings:

(i) If $p \neq 2q$, then a colouring of $P(p, q)$ produces a colouring of $P(p, q-p)$ by the bijection $v_i \mapsto v_{p-i}$, $w_i \mapsto w_{p-i}$ (corresponding to reflecting the drawing about an axis through v_0 and w_0).

(ii) If p and q are coprime, then the inner vertices lie on a p-cycle, and we may redraw the coloured graph with the inner and outer p-cycles interchanged, to produce a colouring of a drawing of $P(p, s)$, where $sq \equiv 1 \pmod p$. That is to say, we use the bijection $v_i \mapsto w_{si},\ w_i \mapsto v_{qi}$.

(iii) If both of the conditions $p \neq 2q$ and p, q coprime apply, then we may obtain a fourth drawing of $P(p, p-s)$, by composition. This corresponds to the fourth bijection of the Klein group, $v_i \mapsto w_{-si},\ w_i \mapsto v_{-qi}$.

We note that these group elements are not always distinct; for example, there are cases where $s = q$.

5.2 Using p-plet colourings

The case $\lambda = r - 1$

Before proceeding to the determination of the fractional chromatic numbers, we show that the p-plet construction enables us to say, for a range of values of r, which of the graphs $P(p, 2)$ have an equable $[r, r-1]$ colouring.

Proposition 5.1 The graph $P(p, 2)$ has a $[(p-1), (p-2)]$ colouring with $N = p, f = 2(p-1)$.

Proof We colour a single p-plet with $p-2$ ◆, one ● and one ■

◆◆ ... ◆●■.

There are $p-2$ outer overlaps and $p-2$ inner overlaps. Each colour occurs $p-1$ times each in outer and inner rings. ■

Proposition 5.2 The graph $P(p, 2)$ has a $[(p-2), (p-3)]$ colouring with $N = p, f = 2(p-2)$.

Proof We colour a single p-plet with $p-3$ ◆, one ●, one ■ and one □:

◆◆ ... ◆●■□.

There are $p-3$ outer overlaps and $p-3$ inner overlaps. Each colour occurs $p-2$ times each in outer and inner rings. ■

Proposition 5.3 The graph $P(p, 2)$ has a $[(p-3),\ (p-4)]$ colouring with

$$N = p, f = 2(p-3), \text{ where } p \neq 6.$$

Proof We colour a single p-plet with $p-4$ ◆, one ●, one ■ and two □:

◆◆ ... ●◆□■□.

There are $p-4$ outer overlaps and $p-4$ inner overlaps. Each colour occurs $p-3$ times each in outer and inner rings. ■

Proposition 5.4 The graph $P(p, 2)$, has no equable $[r, r-1]$ colouring for $r \geq p$.

Proof To colour $P(p, 2)$ with $[p,\ p-1]$ we need $p-1$ ◆, one ● and one ■.

This will entail using more than one p-plet, which will break the sequence

◆◆ ... ◆●

and reduce λ, so that the colouring is impossible. *A fortiori*, a colouring with $r > p$ is impossible. ■

The Fractional Chromatic Numbers of Generalized Petersen Graphs

In this section we work towards a general theorem that gives the fractional chromatic number of a GenPet in terms of the properties of its maximal independent vertex sets (MIVSs). Thus, we now consider colourings with $\lambda = 0$. We continue to work with p-plets; however, since $\lambda = 0$, the p-plets have no ◆ symbol.

We need to translate a p-plet symbol P_i into a description of the corresponding MIVS, M_i. The symbol P_i gives the *colour sets* S_0, T_0, which describe the v_0 and w_0 rows of a particular 'block' of the colouring matrix (such as, for example, the block describing colours 11 to 15 on page 42). In order to find the corresponding M_i, one must read the first column of the block. A rule is needed to describe the MIVSs in terms of the p-plet symbols. In general, if the jth colour of the p-plet is in the colour set S_0, then the $(p-j+1)$th outer vertex, v_{p-j+1}, is in the MIVS (counting modulo p), with a similar rule for T_0 and the inner vertices.

Proposition 5.5 Suppose that, for $i = 1, 2$, $P(p, q)$ has an MIVS, M_i, corresponding to a p-plet P_i with a_i ● and b_i ■ (that is, a_i vertices in the outer cycle and b_i vertices in the inner cycle), where $a_1 < b_1$ and $a_2 > b_2$. Then

$$\chi_f(P(p,q)) \leq \frac{p(b_1 - a_1 + a_2 - b_2)}{b_1 a_2 - b_2 a_1}.$$

Proof Each p-plet P_i produces a non-overlap colouring, where each outer vertex receives a_i colours and each inner vertex receives b_i colours. Thus, the colouring corresponding to $b_1 - a_1$ copies of the p-plet P_2 and $a_2 - b_2$ copies of P_1 has

$(b_1 - a_1)a_2 + (a_2 - b_2)a_1 = b_1 a_2 - b_2 a_1 = (b_1 - a_1)b_2 + (a_2 - b_2)b_1$

colours at each vertex.

We have produced an $[r, 0]$ colouring using N colours, where $r = b_1 a_2 - b_2 a_1$ and $N = p(b_1 - a_1 + a_2 - b_2)$. Thus,

$$\chi_f(P(p,q)) \leq \frac{p(b_1 - a_1 + a_2 - b_2)}{b_1 a_2 - b_2 a_1}, \text{ as required.} \qquad \blacksquare$$

Let us say that an $[r, \lambda]$ colouring using N colours has *efficiency* (see Chapter 1, p 15) $\varepsilon = \frac{r}{N}$. Thus, to find $\chi_f(P(p,q))$, we need to find a colouring of minimum $\frac{N}{r}$ and thus of maximum efficiency. The next proposition considers any $[r, 0]$ colouring of $P(p, q)$, not necessarily equable and not necessarily cyclic, and produces from it a p-plet colouring with the same efficiency.

Proposition 5.6 Let μ be any $[r, 0]$ colouring of $P(p, q)$ using N colours and let μ_i be the colouring achieved by rotating the colours of μ by i positions ($i = 0, \ldots, p-1$).

Then $\rho = \mu_0 + \ldots + \mu_{p-1}$ is a p-plet colouring with the same efficiency as μ.

Proof Consider any particular colour of μ_0; its occurrence in the vertex set can be described by a p-plet P involving □, ● and ■. With its corresponding colours in $\mu_1, \ldots \mu_{p-1}$, it gives rise to the cyclic colouring described by P. Thus, ρ is a sum of p-plet colourings and is a p-plet colouring. Clearly, ρ is a $[pr, 0]$ colouring using pN colours, and so has the same efficiency as μ. ■

Proposition 5.7 There exists a most efficient colouring that is also a p-plet colouring formed *either* by just one p-plet *or* by a combination of just two p-plets as in Proposition 5.5.

Proof By Proposition 5.6, we may find $\chi_f(P(p,q))$ by searching for the most efficient p-plet colouring. Consider such a colouring, formed as above by the p-plets $P_1, \ldots, P_k$, where the p-plet P_i corresponds to an independent vertex set with a_i outer and b_i inner vertices.

Suppose that we take c_i copies of each P_i ($i = 1, \ldots, k$). The resulting colouring has $\sum_{i=1}^{k} c_i a_i$ colours per vertex on the outer vertices and $\sum_{i=1}^{k} c_i b_i$ on the inner vertices. Thus, we require $\sum_{i=1}^{k} c_i a_i = \sum_{i=1}^{k} c_i b_i \ (= r)$ and we wish to maximise the efficiency $\frac{r}{N} = \frac{r}{p\Sigma c_i}$.

If we plot the points (a_i, b_i) on the xy plane, then the point $\frac{(\Sigma c_i a_i, \Sigma c_i b_i)}{\Sigma c_i}$ is in the convex hull H of the (a_i, b_i), and has $\Sigma c_i a_i = \Sigma c_i b_i$ if and only if it lies on the line $y = x$.

Therefore, we maximise the efficiency (hence minimising the estimate of $\chi_f(P(p,q))$) by finding the intersection of the boundary of H with the line $y = x$. This must occur either at a point (a_i, a_i) or on the line segment between two points (a_i, b_i), (a_j, b_j). ■

We must finally consider how the most efficient single p-plet or pair of p-plets may be identified.

The GenPet $P(p, q)$ may have many MIVSs, but from the foregoing analysis, the only relevant information is the numbers of outer and inner vertices. Thus, let $A = \{(a_i, b_i)\}$ be the set of all pairs (a_i, b_i) corresponding to the outer and inner vertices of MIVSs.

If $a_i \le a_j$ and $b_i \le b_j$, at least one of the inequalities being strict, then (a_i, b_i) will *not* contribute to a 'most efficient' colouring, as it does not lie on the boundary of H.

Therefore, we need consider only the set A' of 'winners', a winner being a member (a, b) of A such that, for all i, *either* $a \ge a_i$ *or* $b \ge b_i$.

Then A' can be ordered as $A' = \{(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)\}$, where the a_i are in increasing order and the b_i are in decreasing order.

Theorem 5.8

Case 1 If $a_i < b_i$, $(i = 1, \ldots, k)$, then $\chi_f(P(p,q)) = \frac{p}{a_k}$;

Case 2 If $b_i < a_i$, $(i = 1, \ldots, k)$, then $\chi_f(P(p,q)) = \frac{p}{b_1}$;

Case 3 Otherwise, if $a_i = b_i$ for some i, then $\chi_f(P(p,q)) = \frac{p}{a_i} = \frac{p}{b_i}$;

Case 4 Otherwise, $\chi_f(P(p,q)) = \min\left\{\frac{p(b_i - a_i + a_j - b_j)}{b_i a_j - b_j a_i}\right\}$,

the minimum being taken over all pairs such that $a_i < b_i$, $a_j > b_j$.

Proof

Case 1 If $a_i < b_i$ $(i = 1, \ldots, k)$, then no sum of colourings corresponding to MIVSs will produce a colouring with equal numbers of outer and inner colours. We must instead take a non-maximal IVS with equal such numbers. We therefore choose the p-plet corresponding to an IVS with a_k outer and a_k inner vertices.

Cases 2 and 3 follow similarly, and Case 4 follows from Proposition 4.5. ■

We note that Case 2 does occur. If $p = 3q$, then the inner vertices occur in triangles and $b_1 = \frac{p}{3}$. Thus, $\chi_f(P(p,q)) = 3$, as we expect from the existence of triangles.

Precise Values of χ_f for $q = 2$

$P(p, 2)$ We consider first GenPet graphs in which $q = 2$. We can colour four consecutive vertex pairs (VPs) ●■■●, but, in order to avoid adjacent vertices having identical colours, we colour any five consecutive VPs ●■■●□, and no five consecutive VPs can contain more than four independent vertices. Moreover, for $P(6, 2), \ldots, P(9, 2)$, a p-plet must contain at least two □. Then we have:

$P(5, 2)$ This is Case 3; $\chi_f(P(5,2)) = \frac{10}{4} = \frac{5}{2}$. Clearly, by using the homomorphism from $P(5m, 2)$ to $P(5, 2)$ resulting from 'chaining'as described in Section 5.1, we obtain $\chi_f(P(5m,2)) = \frac{5}{2}$, where m is any positive integer. Moreover, the colouring of $P(5, 2)$ is a [2, 0] colouring, and thus (for any m) $\chi_2(P(5m,2)) = \chi_2(C_5) = 5$, so that by Corollary 4.2, $P(5m, 2)$ has the same overlap profile as C_5.

$P(6, 2)$ This is Case 3. We colour it ●■■●□□, since the 6th VP has both outer and inner vertices adjacent to other vertices. It is convenient to entitle it $P(5 + 1, 2)$.

$P(7, 2)$ This is Case 4. We colour it ●■■●□●□ ●■■●□■□. ($P(5 + 2, 2)$)

$P(8, 2)$ This is Case 3. We colour it ●■■●□●■□. ($P(5 + 3, 2)$)

$P(9, 2)$ This is Case 4. We colour it ●■■●□●■●□ ●■■●□●■■□. ($P(5 + 4, 2)$)

Each of the above colourings may be augmented by inserting the sequence ●■■●, since it is compatible at each end. Colourings of $P(5m, 2)$, $P(5m + 1, 2)$ and $P(5m + 3, 2)$ are thus all Case 3, while $P(5m + 2, 2)$, $P(5m + 4, 2)$ are all Case 4. These colourings consist of p-plets each with the number of □ compatible with the fact that no five consecutive symbols can avoid □. That is to say, a maximum IVS always has $\left\lceil \frac{p}{5} \right\rceil$ □ symbols.

To determine $\chi_f(G)$ in each case, we start from the colouring of the least p and increase $a(G)$ by 2 for each additional 5-plet.

G	$\lvert V(G) \rvert$	$a(G)$	$\chi_f(G)$
$P(5m, 2)$:	$10m$,	$4m$	$\frac{5}{2}$
$P(5m + 1, 2)$*	$10m + 2$	$4m$	$\frac{5m+1}{2m}$
$P(5m + 2, 2)$	$10m + 4$	$4m + 1$	$\frac{10m+4}{4m+1}$
$P(5m + 3, 2)$	$10m + 6$	$4m + 2$	$\frac{5m+3}{2m+1}$
$P(5m + 4, 2)$	$10m + 8$	$4m + 3$	$\frac{10m+8}{4m+3}$

* For $m = 1$, $P(5m+1, 2)$ contains two triangles of inner vertices, corresponding to the fact that $\chi_f(P(6,2)) = \chi_1(P(6,2)) = 3$. By Corollary 4.2, $P(6, 2)$ has the same overlap profile as C_3.

Upper Bounds on χ_f for $q = 3, 4, 5$

$P(p, 3)$ We note first that for all m, s, $\chi_f(P(2m, 2s+1)) = 2$. Thus, all these GenPets are bipartite.

We distinguish three categories of $P(p, 3)$ with p odd, corresponding to $p \equiv 1, 3, 5 \pmod 6$. In each case we construct two p-plets, with $a_1 < b_1$ and $a_2 > b_2$ respectively, in order to apply Proposition 5.5.

(i) These have $a < b$, that is, $|\bullet| < |\blacksquare|$, and are built of a sequence of 6-plets $\blacksquare\blacksquare\blacksquare\bullet\square\bullet$ with endings:

$P(6m+1, 3)$: $\blacksquare\blacksquare\blacksquare\bullet\square\bullet \ldots \blacksquare\blacksquare\blacksquare\bullet\square\bullet \quad \square \qquad |\blacksquare| = 3m, \quad |\bullet| = 2m;$

$P(6m+3, 3)$: $\blacksquare\blacksquare\blacksquare\bullet\square\bullet \ldots \blacksquare\blacksquare\blacksquare\bullet\square\bullet \quad \square\bullet\square \qquad |\blacksquare| = 3m, \quad |\bullet| = 2m+1;$

$P(6m+5, 3)$: $\blacksquare\blacksquare\blacksquare\bullet\square\bullet \ldots \blacksquare\blacksquare\blacksquare\bullet\square\bullet \quad \blacksquare\blacksquare\square\bullet\square \qquad |\blacksquare| = 3m+2, \quad |\bullet| = 2m+1.$

(ii) These have $|\bullet| > |\blacksquare|$:

$\blacksquare\bullet\blacksquare\bullet \ldots\ldots\ldots\ldots \blacksquare\bullet\blacksquare\bullet\square\bullet\square$

$P(6m+1, 3) \quad |\blacksquare| = 3m-1, \quad |\bullet| = 3m;$

$P(6m+3, 3) \quad |\blacksquare| = 3m, \quad |\bullet| = 3m+1;$

$P(6m+5, 3) \quad |\blacksquare| = 3m+1, \quad |\bullet| = 3m+2.$

Then by Proposition 5.5, $\chi_f(P(6m+1,3)) \le \dfrac{(m+1)(6m+1)}{3m^2+2m};$

$$\chi_f(P(6m+3,3)) \le \frac{2(6m+3)}{6m} = \frac{2m+1}{m};$$

$$\chi_f(P(6m+5,3)) \le \frac{(m+1)(6m+5)}{3m^2+4m+2};$$

and, generalizing, for $a \in [1,5]$,

$$\chi_f(P(6m+a,3)) \le \frac{(m+1)(6m+a)}{3m^2+(a-1)m+\frac{a-1}{2}}.$$

We conjecture that these bounds are correct.

We note that, for each $a \in [1,3,5]$, $\lim_{m \to \infty} (\chi_f(P(10m+a,3)) = 2$.

We note also that the smallest odd cycle in $P(6m + 3, 3)$ is of length $2m + 1$, and hence the inequality is an equality:

Proposition 5.9 $\quad \chi_f(P(6m+3,3)) = \frac{2m+1}{m}$. ■

It would be tempting to conclude that $P(6m + 3, 3)$ has the same overlap profile as C_{2m+1}; but since the colouring construction does not give an m-fold colouring but rather a $(3m^2)$-fold colouring, we cannot draw this conclusion. We do, however, conjecture that the overlap profiles are the same.

<u>$P(p, 5)$</u> We consider colourings of $P(p, 5)$ before those of $P(p, 4)$ because the latter present problems that have not occurred in $P(p, 2)$ or $P(p, 3)$. Once again, we construct two p-plets in each case, with $a < b$ and $a > b$ respectively.

We consider only odd values of p, since $\chi_f(P(2m, 5) = 2$ for all m.

(i) Each colouring in which |■| > |●| is based on a sequence of $m - 1$ 10-plets: ●■■■●■●□●■, each with 5■ and 4● , and with a final

	11-plet	●■■■●■●□●□□
	13-plet	●■■■●■●□●□□●□
	15-plet	●■■■●■●□●■□●□●□
	17-plet	●■■■●■●□●■□●■●□●■
or	19-plet	●■■■●■●□●■●■■●□●□●□■

(ii) Each colouring in which |■| < |●| consists of an alternation ●■●■ ... ●■●■●□●□□; a colouring for $10m + a$ VPs consists of $5(m-1)+3$ ■ and $5(m-1)+5$ ●

We represent the maximum independent vertex set in which |■| ≥ |●| by M_1 and that in which |■| ≤ |●| by M_2.

Using these, we construct the table:

	$\lvert M_1\rvert$		$\lvert M_2\rvert$	
p	$\lvert\blacksquare\rvert$	$\lvert\bullet\rvert$	$\lvert\blacksquare\rvert$	$\lvert\bullet\rvert$
$10m+1$	$5m-1$	$4m$	$5m-2$	$5m$
$10m+3$	$5m$	$4m+1$	$5m-1$	$5m+1$
$10m+5$	$5m$	$4m+2$	$5m$	$5m+2$
$10m+7$	$5m+2$	$4m+3$	$5m+1$	$5m+3$
$10m+9$	$5m+3$	$4m+4$	$5m+2$	$5m+4$

There is a general pattern to be described for $P(10m + a, 5)$ ($a = 1, 3, 7, 9$). (We discuss $P(10m + 5, 5)$ separately.) In the MIVS, we note that $\lvert\blacksquare\rvert - \lvert\bullet\rvert = m-1$, and in the alternative IVS ('AIVS'), $\lvert\bullet\rvert - \lvert\blacksquare\rvert = 2$.

Thus, the denominator of $\chi_f(P(p,5)) = (m-1)(\lvert\bullet\rvert \text{ in AIVS}) + 2(\lvert\bullet\rvert \text{ in MIVS})$, and the numerator is $(a+1)p$. Then,

$$\chi_f(P(10m+1,5)) \le \frac{(m+1)(10m+1)}{m(5m+3)}$$

$$\chi_f(P(10m+3,5)) \le \frac{(m+1)(10m+3)}{5m^2+4m+1}$$

$$\chi_f(P(10m+7,5)) \le \frac{(m+1)(10m+7)}{5m^2+6m+3}$$

$$\chi_f(P(10m+9,5)) \le \frac{(m+1)(10m+9)}{5m^2+7m+4}$$

and, generally,

$$\chi_f(P(10m+a,5)) \le \frac{(m+1)(10m+a)}{5m^2+\frac{a+5}{2}m+\frac{a-1}{2}}.$$

<u>$P(10m+5, 5)$</u>

When $m = 1$, $\lvert\bullet\rvert > \lvert\blacksquare\rvert$. The inner vertices lie on triangles, and $\chi_f(P(15,5)) = 3$

When $m = 2$, $\lvert\bullet\rvert = \lvert\blacksquare\rvert$ in M_1 and $\lvert\bullet\rvert > \lvert\blacksquare\rvert$ in M_2. (Case 2). Then

$$\chi_f(P(25,5)) \le \frac{25}{10}.$$

When $m > 2$, $\chi_f(P(10m+5,5)) \le \frac{2m+1}{m}$.

We note that for all $a \in [1,3,5,7,9]$, $\lim_{m\to\infty} (\chi_f(P(10m+a,5)) = 2$. ■

Again, we conjecture that these bounds are correct. We also conjecture that $P(10m+5,5)$

shares the overlap profile of C_{2m+1}.

<u>$P(p, 4)$</u>

Unlike the three colourings for $q = 2$, 3 or 5, there appears to be no general pattern of colouring for $q = 4$.

The minimal graph is $P(9, 4)$. Where two alternative colourings are required to give |■| = |●|, we label the colouring in which |■| > |●| (a), and that in which |■| < |●| (b).

There is a basic 8-plet, which would constitute a degenerate graph, but which can be prefixed to other colour colourings. We label ■●■■●■●□ (a) and ■●■●□●□● (b).

There is also one more version in which |■| = |●|: ■■●■●□●□.

<u>$P(9, 4)$</u> ■■■■●□●□● (a) ■■●■●□●□● (b) $\chi_f(P(9,4)) \leq \frac{18}{7}$.

<u>$P(10, 4)$</u> ■●■■●■●□●□ $\chi_f(P(10,4)) \leq \frac{10}{4}$.

<u>$P(11, 4)$</u> ■●■■●■●□■□● (a) ■●■■●■●□●□● (b) $\chi_f(P(11,4)) \leq \frac{22}{9}$.

<u>$P(12, 4)$</u> ■■■■●□●□●□●□ $\chi_f(P(12,4)) \leq 3$.

(■●■■●■●□●□●□)*Available for chaining as a (b)

<u>$P(13, 4)$</u> ■●■■●■●□■●■●□ (c) ■●■●□■●■●□●□● (d)

$\chi_f(P(13,4)) \leq \frac{39}{16}$.

<u>$P(14, 4)$</u> ■■●■●□■●□■●□■● (a) ■■●■●□■●■●□●□● (b)

$\chi_f(P(14,4)) \leq \frac{28}{11}$.

<u>$P(15, 4)$</u> ■■●■●□■●■■●□●□● $\chi_f(P(15,4)) \leq \frac{15}{6}$.

<u>$P(16, 4)$</u> ■●■■●■●□■●■■●■●□ (a) ■●■■●■●□■●■●□●□● (b)

$\chi_f((P(16,4)) \leq \frac{48}{20}$.

<u>$P(17, 4)$</u> ■●■■●■●□■●■■●□●□● $\chi_f(P(17,4)) \leq \frac{17}{7}$.

<u>$P(18, 4)$</u> ■●■■●■●□■●■■●■●□●□ (c)

■●■■●■●□■●■●□●□●□● (d) $\chi_f(P(18,4)) \leq \frac{54}{22}$.

<u>$P(19, 4)$</u> ■●■■●■●□■●■■●■●□●□● $\chi_f(P(19,4)) \leq \frac{19}{8}$.

<u>$P(20, 4)$</u> $P(10, 4) + P(10, 4)$ $\chi_f(P(20,4)) \leq \frac{20}{8}$

We can use these, by juxtaposing and chaining, to give values of $\chi_f(P(p,4))$. for $p > 20$.

These values are not necessarily minimal. For example, chaining three copies of $P(10, 4)$ suggests $\chi_f(P(30,4)) = \frac{30}{12}$, whereas we find a colouring in which M_1 has 13■ + 12● and M_2 has 7■ + 15●, giving the slightly more economical value $\frac{270}{111}$.

Annex 5.2 is a table of values of $\chi_f(P(p,q))$ for values of (p, q) from (5, 2) to (30, 14).

Annex 5.1

Parameters N, f for Minimal Possible Equable Colourings of Petersen and Generalized Petersen Graphs for Given p, q, r, λ.

		$P(5,2)$		$P(6,2)$		$P(7,2)$		$P(7,3)$		$P(8,2)$		$P(8,3)$		$P(9,2)$		$P(9,3)$		$P(9,4)$	
r	λ	N	f	N	f	N	f	N	f	N	f	N	f	N	f	N	f	N	f
1	0	5	2	3	4	7	2	7	2	4	4	2	8	3	6	3	6	3	6
2	0	5	4	6	4	7	4	7	4	8	4	4	8	6	6	6	6	6	6
2	1	5	4	3	8	7	4	7	4	4	8	4	8	3	12	3	12	3	12
3	0	10	3	9	4	14	3	14	3	8	6	6	8	9	6	9	6	9	6
3	1	5	6	6	6	7	6	7	6	8	6	6	8	9	6	9	6	9	6
3	2	5	6	4	9	7	6	7	6	8	6	4	12	9	6	6	9	9	6
4	0	10	4	12	4	14	4	14	4	16	4	8	8	12	6	12	6	12	6
4	1	10	4	12	4	14	4	14	4	8	8	8	8	9	8	9	8	9	8
4	2	10	4	6	8	7	8	7	8	8	8	8	8	6	12	6	12	6	12
4	3	5	8	6	8	7	8	7	8	X	X	8	8	X	X	9	8	9	8
5	0	25	2	15	4	14	5	14	5	16	5	10	8	15	6	15	6	15	6
5	1	25	2	12	5	14	5	14	5	16	5	10	8	15	6	9	10	18	5
5	2	10	5	12	5	14	5	14	5	16	5	8	10	9	10	9	10	9	10
5	3	10	5	12	5	7	10	7	10	8	10	8	10	9	10	9	10	9	10
5	4	X	X	6	10	7	10	7	10	8	10	8	10	X	X	9	10	X	X
6	0	15	4	18	4	21	4	21	4	16	6	12	8	18	6	18	6	18	6
6	1	15	4	18	4	21	4	21	4	24	4	12	8	18	5	18	6	18	6
6	2	10	6	12	6	14	6	14	6	16	6	12	8	18	6	18	6	18	6
6	3	10	6	9	8	14	6	14	6	12	8	12	8	9	12	9	12	12	9
6	4	10	6	8	9	14	6	14	6	16	6	12	8	9	12	9	12	12	9
6	5	X	X	X	X	7	12	7	12	8	12	8	12	9	12	X	X	9	12
7	0	35	2	21	4	49	2	49	2	28	4	14	8	21	6	21	6	21	6
7	1	35	2	21	4	49	2	49	2	16	74	14	8	18	7	18	7	18	7
7	2	35	2	21	4	49	2	49	2	16	7	14	8	18	7	18	7	18	7
7	3	X	X	12	7	14	7	14	7	16	7	14	8	18	7	18	7	18	7
7	4	10	7	12	7	14	7	14	7	16	7	14	8	18	7	18	7	18	7
7	5	10	7	X	X	X	X	X	X	X	X	X	X	9	14	9	14	9	14
7	6	X	X	X	X	X	X	X	X	8	14	8	14	9	14	9	14	9	14
8	0	20	4	24	4	28	4	28	4	16	8	16	8	24	6	24	6	24	6
8	1	20	4	24	4	28	2	28	2	16	8	16	8	18	8	18	8	18	8
8	2	20	4	24	4	14	8	28	2	16	8	16	8	18	8	18	8	18	8
8	3	20	4	24	4	14	8	14	8	16	8	16	8	18	8	18	8	18	8
8	4	20	4	12	8	28	2	28	2	16	8	16	8	12	12	12	12	12	12
8	5	X	X	12	8	14	8	14	8	16	8	16	8	18	8	18	8	18	8
8	6	10	8	12	8	14	8	14	8	16	8	16	8	X	X	18	8	12	12
8	7	X	X	X	X	X	X	X	X	X	X	X	X	9	16	9	16	9	16
9	0	45	2	27	4	42	3	42	3	24	6	18	8	27	6	27	6	27	6
9	1	30	3	27	4	21	6	21	6	24	6	18	8	27	6	27	6	27	6
9	2	30	3	27	4	21	6	21	6	24	6	18	8	27	6	27	6	27	6
9	3	15	6	18	6	21	6	21	6	24	6	16	9	18	9	18	9	18	9
9	4	15	6	18	6	21	6	21	6	24	6	16	9	18	9	18	9	18	9
9	5	15	6	18	6	21	6	21	6	24	6	16	9	18	9	18	9	18	9
9	6	15	6	12	9	21	6	21	6	24	6	16	9	18	9	18	9	18	9
9	7	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
9	8	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
10	0	50	2	30	4	35	4	35	4	40	4	20	8	36	5	36	5	36	5
10	1	50	2	30	4	35	4	35	4	40	4	20	8	30	5	36	5	36	5
10	2	25	4	24	5	35	4	35	4	40	4	20	8	36	5	18	10	36	5
10	3	25	4	24	5	35	4	35	4	20	8	20	8	36	5	36	5	36	5
10	4	20	5	24	5	35	4	35	4	20	8	16	10	18	10	18	10	18	10
10	5	20	5	15	8	35	4	35	4	20	8	16	10	18	10	18	10	18	10
10	6	20	5	15	8	10	14	10	14	16	10	16	10	18	10	18	10	18	10
10	7	20	5	15	8	10	14	10	14	16	10	16	10	18	10	18	10	18	10
10	8	X	X	12	10	10	14	10	14	16	10	16	10	X	X	X	X	X	X
10	9	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Annex 5.2

Table of Values of $\chi_4(P(p, q))$

q/p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
5	$\frac{5}{2}$																	
6	3																	
7	$\frac{14}{5}$	$\frac{14}{5}$																
8	$\frac{8}{3}$	2																
9	$\frac{18}{7}$	3	$\frac{18}{7}$															
10	$\frac{5}{2}$	2	$\frac{10}{4}$															
11	$\frac{11}{4}$	$\frac{22}{9}$	$\frac{22}{9}$	$\frac{11}{4}$														
12	$\frac{24}{9}$	2	3	2														
13	$\frac{13}{5}$	$\frac{39}{16}$	$\frac{39}{16}$	$\frac{13}{5}$	$\frac{13}{5}$													
14	$\frac{28}{11}$	2	$\frac{28}{11}$	2	$\frac{28}{11}$													
15	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{15}{6}$	3	$\frac{15}{6}$	$\frac{5}{2}$												
16	$\frac{8}{3}$	2	$\frac{48}{20}$	2	$\frac{32}{13}$	2												
17	$\frac{34}{13}$	$\frac{51}{22}$	$\frac{17}{7}$	$\frac{17}{7}$	$\frac{51}{22}$	$\frac{17}{7}$	$\frac{34}{13}$											
18	$\frac{18}{7}$	2	$\frac{18}{7}$	2	3	2	$\frac{18}{7}$	2										
19	$\frac{38}{15}$	$\frac{76}{33}$	$\frac{19}{8}$	$\frac{19}{8}$	$\frac{76}{33}$	$\frac{38}{15}$	$\frac{38}{15}$	$\frac{38}{15}$										
20	$\frac{5}{2}$	2	$\frac{10}{4}$	2	$\frac{20}{8}$	2	$\frac{5}{2}$	2										
21	$\frac{21}{8}$	$\frac{21}{9}$	$\frac{63}{26}$	$\frac{63}{26}$	$\frac{42}{17}$	3	$\frac{42}{17}$											
22	$\frac{44}{17}$	2	$\frac{22}{9}$	2	$\frac{44}{18}$	2	$\frac{22}{9}$	2										
23	$\frac{23}{9}$	$\frac{92}{41}$	$\frac{23}{9}$	$\frac{69}{29}$	$\frac{23}{18}$	$\frac{46}{19}$	$\frac{92}{41}$	$\frac{69}{29}$	$\frac{46}{19}$	$\frac{23}{9}$								
24	$\frac{48}{19}$	2	$\frac{48}{19}$	2	$\frac{24}{10}$	2	3	2	—	2								
25	$\frac{5}{2}$	$\frac{125}{56}$	$\frac{25}{10}$	$\frac{5}{2}$		$\frac{50}{21}$	$\frac{125}{56}$											
26	$\frac{13}{5}$	2	$\frac{32}{21}$	2	$\frac{26}{11}$	2	$\frac{26}{10}$	2		2								
27	$\frac{54}{21}$	$\frac{27}{12}$	$\frac{27}{11}$	$\frac{81}{35}$		$\frac{54}{26}$	$\frac{27}{12}$	3	$\frac{27}{12}$	$\frac{81}{35}$		$\frac{54}{21}$						
28	$\frac{28}{11}$	2	$\frac{28}{11}$	2		2	$\frac{28}{12}$	2		2		2						
29	$\frac{58}{23}$	$\frac{145}{56}$		$\frac{87}{38}$	$\frac{87}{38}$	$\frac{58}{28}$			$\frac{145}{56}$				$\frac{58}{23}$					
30	$\frac{5}{2}$	2		2		2		2	3	2		2						
31	$\frac{31}{12}$	$\frac{186}{85}$	$\frac{62}{25}$	$\frac{124}{54}$	$\frac{124}{54}$	$\frac{217}{93}$	$\frac{62}{25}$	$\frac{217}{93}$	$\frac{186}{85}$					$\frac{31}{12}$				
32	$\frac{64}{25}$	2		2		2		2		2		2		2				
33	$\frac{33}{13}$	$\frac{33}{15}$		$\frac{132}{58}$						3		$\frac{132}{58}$			$\frac{33}{13}$			
34	$\frac{68}{27}$	2		2		2		2		2		2		2				
35	$\frac{5}{2}$	$\frac{210}{97}$		$\frac{5}{2}$							$\frac{210}{97}$					$\frac{5}{2}$		
36	$\frac{36}{14}$	2		2		2		2		2	3	2		2		2		
37	$\frac{74}{29}$	$\frac{259}{120}$		$\frac{148}{66}$							$\frac{210}{97}$			$\frac{148}{66}$			$\frac{74}{29}$	
38	$\frac{38}{15}$	2		2		2		2		2		2		2		2		
39	$\frac{78}{31}$	$\frac{39}{18}$		$\frac{156}{70}$			$\frac{156}{70}$					3						$\frac{78}{31}$
40	$\frac{5}{2}$	2		2		2		2		2		2		2		2		2

Chapter 6: Complete Graphs K_n

Finding precise values for all overlap chromatic numbers for all complete graphs appears to be an intractable problem, since such a determination would solve many existence problems in design theory. Constructions for *balanced incomplete block designs* (BIBDs), however, translate into constructions for equable overlap colourings of complete graphs. An argument in this chapter shows that, when such a colouring of a complete graph is equable, then this actually gives the exact overlap number; moreover, juxtapositions of such colourings give upper bounds for overlap chromatic numbers. The chapter concludes by referring to a body of work on *constant-weight codes* that translates into lower and upper bounds for these overlap numbers.

6.1 BIBDs and exact determinations of overlap chromatic numbers

The *CRC Handbook of Combinatorial Designs* [16] page 25 gives this definition:

'A *balanced incomplete block design* (BIBD) ia a pair (V, B) where V is a v-set and B is a collection of b k-subsets of V (*blocks*) such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. The numbers v, b, r, k, λ are *parameters* of the BIBD.'

We now describe an equivalence between BIBDs and equable overlap colourings of complete graphs.

Proposition 6.1 There is an equivalence between equable overlap colourings μ of the complete graph K_n, with parameters $[n, r, \lambda, N, f]$, and BIBDs D with parameters $[v, b, r, k, \lambda]$, obtained by interpreting the vertex set of K_n as the set of varieties of D and the colour set as the set of blocks of D (so that the block corresponding to the colour c is the set of vertices v such that $c \in \mu(v)$). Thus, overlap parameters correspond to BIBD parameters if we re-order the latter as v, r, λ, b, k).

Proof Let μ be such a colouring of K_n. Thus μ describes an incidence structure between $V(K_n)$ and the colour set $[N]$, which may be interpreted as a design

$D = (V, B)$ in which $V = V(K_n)$ and $B = [N]$, each block c consisting of the set of vertices having c as one of their colours. Then, the number r of colours at each vertex is the number of blocks to which each variety belongs and is the replication of D while f (the number of times each colour is used in the colouring) corresponds to the size k of each block. Finally, λ represents, for each pair (v, w) of vertices of K_n, the number of colours present at v and w and hence (in D) the number of blocks contsining each pair of varieties. Conversely, every balanced block design D may be derived from an overlap colouring in this way. ■

It is clear that $\chi_{r,0}(K_n) = rn$ and $\chi_{r,r}(K_n) = r$; we shall refer to the $[r, 0]$ and $[r, r]$ colourings of any complete graph as *trivial.*

In discussing a specific K_n, we represent the colouring as $[n, r, \lambda, N, f]$. (This does not imply that there is only one colouring with these parameters.) As we shall now show, if n, r, and λ are given, N and f are uniquely determined, and it is often sufficient to give a colouring as $[n, r, \lambda]$.

Proposition 6.2 For any equable colouring of a complete graph, K_n,

$$N = \frac{nr^2}{(n-1)\lambda + r}. \qquad (1)$$

Proof This follows from Proposition 1.6, since a complete graph has universal vertices. ■

By substituting $N = \frac{nr}{f}$ in (1), we arrive at the further expression:

$$f = \frac{(n-1)\lambda + r}{r}, \qquad (2)$$

from which we may immediately derive $\lambda(n-1) = r(f-1)$, corresponding to a well-known expression for BIBDs, (see [16], page 25). ■

It is convenient to refer to the following 'glossary':

<u>*K_n Parameters*</u>	n	r	λ	N	f
<u>*Design Theory Parameters*</u>	v	r	λ	b	k

In view of this equivalence between equable graph colourings and BIBDs, we add to our list of expressions the equivalent of Fisher's Inequality:

In a BIBD, $b \geq v$ ([16], page 26), which becomes $N \geq n$.

A BIBD in which $b = v$ is said to be *symmetric* ([16] page 26); by analogy, we shall say that a colouring of K_n with $N = n$ is *symmetric*.

Two or more colourings of K_n can be juxtaposed as in Chapter 1. If the colourings in the juxtaposition all have the same frequency f, then the juxtaposition is equable. We now show that for given $[n, r, \lambda]$, a juxtaposition using colourings with different frequencies requires more colours than than an equable colouring if it exists. (For example, there is an equable colouring of K_{15}: [15, 14, 4, 42, 5] and a non-equable [14, 4] colouring obtained by juxtaposing the two equable colourings [15, 7, 1, 35, 3] and [15, 7, 3, 15, 7], which requires 50 colours in comparison with the 42 required by the equable colouring.)

Proposition 6.3 For given r, λ, an equable colouring of complete graph K_n is more economical than a juxtaposition of equable colourings of K_n with different frequencies.

Proof Dividing expression (1) by r, we may express the formula for N in an equable colouring $[n,r,\lambda,N,f]$ colouring μ of K_n as

$$\frac{N}{r} = \frac{n}{(n-1)\frac{\lambda}{r}+1}.$$

Let $\frac{\lambda}{r} = x(K_n,\mu) = x$; $\frac{N}{r} = y(K_n,\mu) = y$ (as in page14), then $y = \frac{n}{(n-1)x+1}$, and the rational parameters (x, y) for any equable colouring of K_n lie on a hyperbola whose sense is convex downwards.

Now, for $i = 1, ..., q$ let μ_i be an equable colouring $[n, r_i, \lambda_i, N_i, f_i]$ of K_n and let $M = \sum_{i=1}^{q} \mu_i$ be the juxtaposition of the μ_i (where not all the f_i are equal).

Suppose that M is also an $[r,\lambda]$ colouring; that is, $r = r_1 + ... + r_q$ and $\lambda = \lambda_1 + ... + \lambda_q$. Let

$$X = x(K_n,M) = \frac{\lambda_1 + ... + \lambda_q}{r_1 + ... + r_q} = \frac{r_1x_1 + ... + r_qx_q}{r_1 + ... + r_q} (= x(K_n,\mu)) \text{ and}$$

$$Y = y(K_n,M) = \frac{N_1 + ... + N_q}{r_1 + ... + r_q} = \frac{r_1y_1 + ... + r_qy_q}{r_1 + ... + r_q}.$$

Thus (X, Y) is a convex linear combination of the (x_i,y_i) and must thus lie within the hyperbola and therefore above the point (x, y); that is, $N_1 + ... + N_q > N$. ∎

Corollary 6.4 If there is an equable colouring $[n,r,\lambda,N,f]$ of K_n, then

$$\chi_{r,\lambda}(K_n) = \frac{nr^2}{(n-1)\lambda + r}.$$

Proof Let μ be this colouring, and suppose that there is a non-equable $[r,\lambda]$ colouring θ of K_n, using Q colours. Suppose these colours take distinct frequencies $f_1, ..., f_q$. For each $i = 1, ..., q$ let θ_i be the colouring that uses the colours of θ that have frequency f_i. Thus, θ is the juxtaposition of the θ_i. These colourings may not individually have constant r or λ, and thus may not be true overlap colourings, but their juxtaposition is θ. Moreover, if Θ_i is the juxtaposition of the images of θ_i under all permutations of the vertices, then each Θ_i is a true overlap colouring, with frequency f_i. Let M be the juxtaposition of all permutation images of μ; then M is an equable $[n!r,n!\lambda]$ colouring of K_n requiring $n!N$ colours, while the juxtaposition Θ of the Θ_i requires $n!Q$ colours. By the proposition, $Q > N$.

Thus, $N = \chi_{r,\lambda}(K_n)$. ∎

The *Handbook of Combinatorial Designs* has a table ([16], pages 36-58) of BIBD parameters, whose first five columns are the parameters v, b, r, k, λ and whose sixth column indicates a lower bound for the number of BIBDs with these parameters. Thus, whenever this number is non-zero, Corollary 6.4 allows the conclusion that $\chi_{r,\lambda}(K_v) = b$.

Limitations arising from Prime Parameters

We now consider some limitations on N that arise if one or more of the parameters of a colouring is prime. Propositions 6.5 - 6.7 concern the parameters of colourings that are assumed to be equable and non-trivial.

Proposition 6.5 The parameters n and N have a common factor. Thus, if n is prime then N is a multiple of n.

Proof We note that $\frac{nr}{N} = f$ is an integer and that $N \geq r$. Thus, $N = r$ would give a trivial colouring. Therefore $N > r$, and so N and n have a common factor. ■

Proposition 6.6 If one of $n-1$, r is prime, then it divides the other.

Proof We make use of the relation $\lambda(n-1) = r(f-1)$.

<u>*$n-1$ prime*</u> Since $\lambda = \dfrac{r(f-1)}{n-1}$, and $f-1 < n-1$, it follows that $(n-1)$ divides r.

<u>*r prime*</u> Since $\lambda < r$ and $f-1 = \dfrac{\lambda(n-1)}{r}$, it follows that r divides $(n-1)$. ■

Proposition 6.7 If n, r are both prime, then the colouring is symmetric

and $n = c^2\lambda + c + 1$, for some integer c.

Proof Since n is prime, $N = an$, by Proposition 6.5 above, so that $r = af$.

Since r is prime, and f is an integer, either $a = r, f = 1$, which gives the trivial $[r, 0]$ colouring, or $a = 1$. Then $N = n$; the colouring is symmetric.

Substituting in (1), and multiplying out,

$$r^2 = (n-1)\lambda + r, \text{ so } r(r-1) = \lambda(n-1).$$

Since $\lambda < r$, $n-1$ is a multiple of r, say $n-1 = cr$, and $r-1 = c\lambda$.

So $\qquad r = c\lambda + 1$ and $n-1 = cr$, and $n = c^2\lambda + c + 1$. ■

For given n and r, we can determine an upper bound of equable colourings.

Since $N \geq n > f$, $N > f$, it follows that $1 < f < \sqrt{nr}$. Also, $\lambda = \dfrac{r(f-1)}{n-1}$.

If $n-1$ is prime, then, since $f < n$, $(n-1)$ divides r.

If $n-1$ is not prime, then, if we reduce $\frac{r}{n-1}$ to its lowest terms, $f-1$ must be a multiple of the new denominator. Some examples make this clear:

(a) $\quad n = 10, r = 9.\ \ 1 < f \leq 9.$ Since f divides $90, f = 2, 3, 5, 6$ or 9.

$$\lambda = \frac{9(f-1)}{9} = f-1; \quad \text{then} \quad \lambda = 1, 2, 4, 5 \text{ or } 8.$$

There are at most five parameter sets.

(b) $n = 10, r = 6$. $1 < f \leq 7$. Since f divides 60, $f = 2, 3, 4, 5$ or 6;

$$\lambda = \frac{6(f-1)}{9} = \frac{2(f-1)}{3}; \quad \text{then } f = 4, \lambda = 2.$$

There is only one parameter set.

Colouring Constructions

The above results do not describe the constructions of the BIBDs (or equivalently, equable colourings of complete graphs) that give rise to these parameters. The constructions which follow, most of which are based on well-known constructions for BIBDs, are, however, of some interest in their own right, and we now describe these; we give appropriate references.

A process which we call *colour section* can be applied to known symmetric colourings (in which $n = N$ and, consequently, $r = f$). It consists of deleting all vertices in which a given colour occurs, thus reducing n by r and N by 1, keeping r and λ unchanged. As for the effect on frequency,

$$f - f' = \frac{(n-1)\lambda + r}{r} - \frac{(n-r-1)\lambda + r}{r} = \lambda.$$

Thus, in general, the parameters of the second colouring are:

$$n' = n - r,\ r' = r,\ \lambda' = \lambda,\ N' = N - 1,\ f' = f - \lambda.$$

A similar process, which we call *colour intersection*, can also be applied to symmetric colourings. It consists of deleting all vertices in which a given colour does not appear, and then deleting that colour from each of the remaining vertices, producing the *derivative*. In this case,

$$n' = r;\ r' = r - 1;\ \lambda' = \lambda - 1;\ N' = N - 1;\ f' = f$$

These constructions correspond respectively to what in [17] are called *variety section* and *intersection*, and the resulting colourings correspond to the *residual* and *derived* designs as described on page 25 of [16].

The definition of complement of a colouring in Chapter 1 corresponds to the definition of complement on page 26 of [16]. This may be the same as the original, or different, and has

$$n' = n;\ r' = N - r;\ \lambda' = N - 2r + \lambda.$$

Constructions and Overlap Parameters

A number of constructions follow standard patterns, and we now discuss these, where possible with algorithms that enable us to find feasible colourings. There are in many cases alternative colourings and alternative algorithms, but, since our intention is to find feasible colourings, we have in general limited ourselves to one algorithm.

(In this chapter, for compactness, we represent vertices in columns and colours in rows.)

Proposition 6.8 $\chi_{r,\lambda}(K_n) = \binom{n}{a}$ whenever $r = \binom{n-1}{a-1}, \lambda = \binom{n-2}{a-2}, 1 \le a \le n$, (where $\binom{n-2}{-1}$ is interpreted as 0).

Proof This follows from Corollary 6.4 by using the BIBD whose blocks consist of all a-sets of $\{1, \ldots, n\}$. ■

Corollary 6.9

(a) $\chi_{n-1,n-2}(K_n) = n.$

Construction We assign $N = n$ colours to each vertex and delete a distinct colour from each.

(b) $\chi_{n-1,1}(K_n) = \binom{n}{2}$

Construction A recursive algorithm for this is

$$K_2\!:\ \begin{matrix}1 & 1\end{matrix} \qquad K_3\!:\ \begin{matrix}1 & 1 & 2\\ 2 & 3 & 3\end{matrix} \qquad K_4\!:\ \begin{matrix}1 & 1 & 2 & 4\\ 2 & 3 & 3 & 5\\ 4 & 5 & 6 & 6\end{matrix} \qquad \ldots\ldots\ldots\ldots$$

Proposition 6.10 Let s be a prime power, $1 \le i \le d$ and

$$r = \frac{(s^d - 1)(s^{d-1} - 1)\dots(s^{d-i+1} - 1)}{(s^i - 1)(s^{i-1} - 1)\dots(s-1)}, \lambda = \frac{(s^{d-1} - 1)\dots(s^{d-i+1} - 1)}{(s^{i-1} - 1)\dots(s-1)}.$$

(a) if $n = s^d + s^{d-1} + \dots + 1$, then $\chi_{r,\lambda}(K_n) = \dfrac{(s^{d+1} - 1)(s^d - 1)\dots(s^{d-i+1} - 1)}{(s^{i+1} - 1)(s^i - 1)\dots(s-1)}$;

(b) if $n = s^d$, then $\chi_{r,\lambda}(K_n) = s^{d-i} r$.

Proof This follows from Corollary 6.4 by using the BIBDs of Propositions 2.36 and 2.37 ([16], pp 705, 706). These consist, respectively, of the i-dimensional subspaces of the d-dimensional projective geometry and of the d-dimensional affine geometry, on the Galois field of order s. ■

Choosing $i = 1, d = 2$ gives the following corollary:

Corollary 6.11 If s is a prime power, then:

(a) $\chi_{s+1,1}(K_{s^2+s+1}) = s^2 + s + 1$,

(b) $\chi_{s+1,1}(K_{s^2}) = s(s+1)$.

Construction An example of (a) is K_{13}, $s = 3$:

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*
1	1	1	1	2	2	2	3	3	3	4	4	4
2	5	8	11	5	6	7	5	6	7	5	6	7
3	6	9	12	8	9	10	10	8	9	9	10	8
4	7	10	13	11	12	13	12	13	11	13	11	12

The complement is $[s^2 + s + 1, s^2, s^2 - s, s^2 + s + 1, s^2]$, also symmetric.

The residual is $[s^2, s + 1, 1, s^2 + s, s]$. ■

Construction An example of (b) is K_9, $s = 3$

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	1	1	2	2	2	3	3	3
4	7	10	4	5	6	4	5	6
5	8	11	7	8	9	8	9	7
6	9	12	10	11	12	12	10	11

Construction We can generalize to higher dimensions, giving the feasible parameter sets

$$\left[\frac{s^p - 1}{s-1}, \frac{s^{p-1} - 1}{s-1}, \frac{s^{p-2} - 1}{s-1}, \frac{s^p - 1}{s-1}, \frac{s^{p-1} - 1}{s-1}\right]$$

The complement has parameters $[\frac{s^p-1}{s-1}, s^p, s^{p-2}(s-1), \frac{s^p-1}{s-1}, s^p]$
The residual has parameters $[s^{p-1}, \frac{s^{p-1}-1}{s-1}, \frac{s^{p-2}-1}{s-1}, \frac{s^p-s}{s-1}, s^{p-2}]$.

The derivative has parameters $[\frac{s^{p-1}-1}{s-1}, \frac{s^{p-1}-s}{s-1}, \frac{s^{p-2}-s}{s-1}, \frac{s^p-s}{s-1}, \frac{s^{p-2}-1}{s-1}]$. ■

Proposition 6.12 Let $n = 4t+3$ be a prime power. Then

$$\chi_{2t+1,\lambda}(K_{4t+3}) = 4t+3$$

Proof This follows from Corollary 6.4 by using the BIBD of Corollary 2.1.7 in [2, p 43], which asserts the existence of a symmetric $(4t+3, 2t+1, t)$ design. ■

Construction We assign all non-zero squares modulo n to one vertex of our graph, and then cycle through the remaining $n-1$ vertices to give a colouring for K_n:

$$[n, \frac{n-1}{2}, \frac{n-3}{4}, n, \frac{n-1}{2}].$$

A simple example is K_7. The non-zero squares modulo 7 are 1, 2, 4, and we colour K_7:

1	*2*	*3*	*4*	*5*	*6*	*0*		v_0	v_1	v_2	v_3	v_4	v_5	v_6
1	2	3	4	5	6	0	or, with our convention:	0	1	2	3	4	5	6
2	3	4	5	6	0	1		1	2	3	4	5	6	0
4	5	6	0	1	2	3		3	4	5	6	0	1	2

If we consider the $n-1$ non-zero integers modulo n, we note that

$$(n-m)^2 \equiv m^2 \pmod{m}; \quad \text{thus } r = \frac{n-1}{2}.$$

This gives a colouring with parameters $\left[n, \frac{n-1}{2}, \frac{n-3}{4}, n, \frac{n-1}{2}\right]$.

The parameters of the colouring of the complement are $\left[n, \frac{n+1}{2}, \frac{n+1}{4}, n, \frac{n+1}{2}\right]$.

The parameters of the colouring of the residual are $\left[\frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, n-1, \frac{n+1}{4}\right]$.

■

Proposition 6.13 Let $n = 4t+3$. Then

$$\chi_{4t+3,2t+1}(K_{4t+4}) = 8t+6.$$

Proof The proof follows that of Corollary 6.3.5 of [2, p 139].

Let μ be the colouring of K_{4t+3} of Proposition 6.1.2. First take a juxtaposition of two copies of μ (using disjoint colour sets S_1, S_2). Next, adjoin a new vertex and allocate to it the colour set S_2. Finally, to each of the original vertices add one colour from S_1 distinct from those already present at that vertex. The result is an equable colouring with the required parameters. ■

Construction The following example is of $\chi_{7,3}(K_8)$:

Vertex	v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7
	0	1	2	3	4	5	6	7
	1	2	3	4	5	6	0	8
	3	4	5	6	0	1	2	9
	7	8	9	10	11	12	13	10
	8	9	10	11	12	13	7	11
	10	11	12	13	7	8	9	12
	6	0	1	2	3	4	5	13

This is a colouring with parameters $[4t+4, 4t+3, 2t+1, 8t+6, 2t+2]$. The colouring is self-complementary. ■

A further procedure for finding colourings consists of colouring n vertices with r colours at each, with no overlap, and taking the complement. As a simple example, with $n = 4, r = 2$, we can devise a colouring with 6 colourings at each vertex, with overlap 4.

Generally: $n' = n;\ r' = nr - r;\ \lambda' = n - 2r;\ N' = nr;\ f' = n - 1.$

Relations Between Colourings of Complete Graphs and Codes

A *binary code of length N* is a set of strings of N 0s and 1s. The connection with colourings is that if $\{1, ..., N\}$ is a palette of colours, then the 1s in any particular string (a *codeword*) identify a subset of the palette, which can be thought of as a set of colours allocated to a vertex of a graph. The *Hamming distance* of two codewords **u**, **v**, denoted by d(**u**, **v**), is the number of places among the N where they differ. Consider an $[r, \lambda]$ overlap colouring of K_n with a palette of N colours. Each vertex is now associated with a codeword having r 1s and $(N-r)$ 0s, and the colouring is a code of length N, constant weight r and size n; moreover, the Hamming distance of any two codewords is $2(r-\lambda)$.

We exhibit a relation between the parameters of some overlap colourings and code bounds. The definitions and terms that we use to describe constant-weight codes are those used by [1], which we quote:

'An (n, d, w) constant-weight binary code is a set of binary vectors of length n, such that each vector contains w ones and $n-w$ zeros, and any two vectors differ in at least d positions.'

The distance d of two codewords **u** and **v** implies that there are $\delta = \frac{d}{2}$ codeword positions in which **u** has 1 and **v** has 0, and a further $\delta = \frac{d}{2}$ positions in which **u** has 0 and **v** has 1. Thus, the number of positions in which both have 1 is $w-\delta$. The weight w of a constant-weight binary code thus corresponds to the number of colours r at a vertex of the related complete graph, and $(w-\delta)$ to the overlap λ.

Constant-weight codes of *minimum* Hamming distance have been studied, and tables exist giving bounds for $A(N, d, w)$, the largest size of a binary code of length N, minimum distance d and constant weight w. In 1990, [4] produced lower bounds for $d \leq 14$, $N \leq 28$ (with certain values indicated as exact values); in 2000, [1] produced similar tables giving upper and lower bounds; and in 2006 [16] extended the results to $d \leq 14$, $N \leq 63$. The connection between these bounds and bounds on $\chi_{r,\lambda}(K_n)$ is as follows.

The existence of a code of length N, minimum distance $d = 2\delta$, weight w and size n is not immediately informative, since the code may not have constant distance. However, the *non-existence* of such a code implies that there is no $[w, w-\delta]$ colouring of K_n with palette size N, so that $\chi_{w,w-\delta}(K_n) > N$. That is, $A(N, 2\delta, w) < n$ implies $\chi_{w,w-\delta}(K_n) > N$. Conversely, if $\chi_{w,w-\delta}(K_n) \leq N$, then a code of length N, weight w and size n and distance (hence minimum distance) 2δ exists, and so $A(N, 2\delta, w) \geq n$. Therefore an upper bound of $n-1$ on

$A(N, 2\delta, w)$ implies a lower bound of $N+1$ on $\chi_{w,w-\delta}(K_n)$, and conversely an upper bound of N on $\chi_{w,w-\delta}(K_n)$ implies a lower bound of n on $A(N, 2\delta, w)$.

We display, as an example, an extract from Table IV on page 2393 of [1]. We change the 'n' in that table to 'N', our equivalent.

Table 1

Values of $A(N, 10, w)$

N	w ($w-5$)								
	6 (1)	7 (2)	8 (3)	9 (4)	10 (5)	11 (6)	12 (7)	13 (8)	14 (9)
12	2	(2)	(1)	(1)	(1)	(1)	(1)	(1)	(1)
13	2	(2)	(2)	(1)	(1)	(1)	(1)	(1)	(1)
14	2	2	(2)	(2)	(1)	(1)	(1)	(1)	(1)
15	3	3	(3)	(3)	(3)	(1)	(1)	(1)	(1)
16	3	4	4	(4)	(3)	(3)	(1)	(1)	(1)
17	3	5	6	(6)	(5)	(3)	(3)	(1)	(1)
18	4	6	9	10	(9)	(6)	(4)	(3)	(1)
19	4	8	12	19	(19)	(12)	(8)	(4)	(3)
20	5	10	17	20	38	(20)	(17)	(10)	(5)
21	7	13	21	27-35	38-42	(38 – 42)	(27 – 35)	(21)	(13)

Here, since $d = 10, \delta = 5$, and we have added to the table as printed the value of $(w - \delta)$.

We have completed this table in accordance with equations 35, 37 and 38 of Theorem 8, page 2378 of [1]. The tables in [1] tabulate $A(N, d, w) = A(N, 2\delta, w)$ only when $N \geq d+2$ and $\frac{d}{2} < w \leq \frac{N}{2}$. If $N \leq d+2$, then clearly $A(N, d, w) \leq 2$, and it is straightforward to determine these values. The entries which complete the table are bracketed.

Consider an entry in Table 1; say, the entry 3 in the row $N = 16$ and the column $w = 6$. This asserts that there is a set of 3 codewords of length 16 having constant weight 6, with minimum distance 10 between any pair; and this is the largest such set. Now, the colour sets of a [6, 1] colouring of K_4 using 16 colours would provide a set of 4 such codewords, contradicting the table; thus, this entry asserts that $\chi_{6,1}(K_4) > 16$. Similarly, $\chi_{6,1}(K_4) > 17$, and the table provides the information that $\chi_{6,1}(K_4) \geq 18$.

Next, consider $\chi_{6,1}(K_6)$; the entry in the row for $N = 20$ asserts that $\chi_{6,1}(K_6) > 20$, and so we conclude that $\chi_{6,1}(K_6) \geq 21$. Thus, each lower bound on an overlap parameter is given by the *least* value of N whose row entry is *at least* equal to the order of the complete graph whose parameter is to be bounded.

Table 2

Lower and Upper Bounds for some $\chi_{r,\lambda}(K_n)$

n	$[r, \lambda]$			
	[6, 1]	[7, 2]	[8, 3]	[9, 4]
3	15 15	15 15	15 15	15 15
4	18 18	16 16	16 16	16 17
5	20 20	17 17	17 17	17 17
6	21 21	18 –	17 18	17 18
7	**21**	19 –	18 18	18 18
8		19 –	18 18	18 18
9		20 –	**18**	18 18
10		20 –		**18**

In Table 2, we display lower bounds of some $\chi_{r,\lambda}(K_n)$, derived from Table 1 and some upper bounds found by trial; the actual colourings for the latter are shown below. Where both lower and upper bounds are known, the lower are on the left and the upper on the right. Emboldened entries are of equable colourings. Other single entries are from Table 1. In order to relate the two tables, we note that the first occurrence of w in Table 1 becomes n in Table 2. The quantity $w - \delta$ corresponds to our λ.

Inserting entries in the [7, 2] column proved difficult, though we have found colourings in agreement for the first three entries (see below). We speculate that this problem may be related to the fact that there are no equable colourings for any K_n [7, 2], as shown by the table in [10].

We shall now verify the upper bounds by giving lists of explicit colourings.

$[r,\lambda] = [6,1]$

K_3

1	1	2
2	3	3
4	5	6
7	8	9
10	11	12
13	14	15

K_4

1	1	2	4
2	3	3	5
4	5	6	6
7	8	9	10
11	12	13	14
15	16	17	18

K_5

1	1	2	4	7
2	3	3	5	8
4	5	6	6	9
7	8	9	10	10
11	12	13	14	15
16	17	18	19	20

K_6

1	1	2	4	7	11
2	3	3	5	8	12
4	5	6	6	9	13
7	8	9	10	10	14
11	12	13	14	15	15
16	17	18	19	20	21

K_7

1	1	2	4	7	11	16
2	3	3	5	8	12	17
4	5	6	6	9	13	18
7	8	9	10	10	14	19
11	12	13	14	15	15	20
16	17	18	19	20	21	21

$[r,\lambda] = [7,2]$

K_3

1	1	3
2	2	4
3	8	8
4	9	9
5	10	13
6	11	14
7	12	15

K_4

1	1	3	5
2	2	4	6
3	8	8	10
4	9	9	11
5	10	13	13
6	11	14	14
7	12	15	16

K_5

1	1	3	5	1
2	2	4	6	7
3	8	8	10	12
4	9	9	11	14
5	10	13	13	15
6	11	14	14	16
7	12	15	16	17

$[r,\lambda] = [8,3]$

K_3

1	1	4
2	2	5
3	3	6
4	9	9
5	10	10
6	11	11
7	12	14
8	13	15

K_4

1	1	4	6
2	2	5	7
3	3	6	8
4	9	9	11
5	10	10	12
6	11	11	13
7	12	14	14
8	13	15	16

K_5

1	1	4	6	1
2	2	5	7	4
3	3	6	8	7
4	9	9	11	9
5	10	10	12	12
6	11	11	13	15
7	12	14	14	16
8	13	15	16	17

K_9

1	1	1	1	2	3	3	4	5
2	2	2	5	5	4	6	6	7
3	3	4	6	7	8	7	8	8
4	9	9	10	10	10	11	9	9
5	10	14	11	12	13	13	11	12
6	11	15	14	16	14	15	12	13
7	12	16	15	17	16	16	17	14
8	13	17	18	18	18	17	18	15

$[r, \lambda] = [9, 4]$

K_3			K_4				K_5				
1	1	5	1	1	5	1	1	1	5	1	3
2	2	6	2	2	6	2	2	2	6	2	4
3	3	7	3	3	7	8	3	3	7	8	5
4	4	8	4	4	8	9	4	4	8	9	6
5	10	10	5	10	10	10	5	10	10	10	12
6	11	11	6	11	11	12	6	11	11	12	14
7	12	12	7	12	12	15	7	12	12	15	15
8	13	13	8	13	13	16	8	13	13	16	16
9	14	15	9	14	15	17	9	14	15	17	17

The [6, 1] colourings of K_n ($3 \le n \le 6$) have each been found by juxtaposing an equable $[n-1, 1]$ colouring with a $[7-n, 0]$ colouring (italicized). They could also have been arrived at by deleting $7-n$ columns from the equable [6, 1] colouring of K_7 in the list.

For the tables set out in [1], wherever a corresponding equable colouring exists, the number of colours required corresponds to the entry in their table. We list these:

Table I	$K_7[3, 1]$	
Table II	$K_6[5, 2]$	
Table III	$K_7[6, 2]^*$; $K_8[7, 3]$; $K_{10}[6, 2]$	* see Table I.
Table IV	$K_9[8, 3]$; $K_{10}[9, 4]$	
Table V	$K_8[7, 1]$; $K_9[8, 3]$; $K_{13}[8, 2]$.	

Chapter 7: Graphical Representation

Our use in this chapter of the capitalized form 'Graph' is specifically to refer to a plot in the (x, y) plane, with x and y as abscissa and ordinate.

It is convenient, both practically and theoretically, to display the fractional parameters of the overlap colourings of a graph G graphically, by displaying the region in the (x, y) plane corresponding to the possible parameters (x, y) of any colouring of G.

Let $\mu_1, \ldots, \mu_k$ be any sequence of overlap colourings of G, such that each μ_i has integer parameters r_i, λ_i, N_i and hence fractional parameters $(x_i, y_i) = \left(\frac{\lambda_i}{r_i}, \frac{N_i}{r_i}\right)$. Then the juxtaposition $\mu = \sum_i \mu_i$ has integer parameters $r = \sum_i r_i, \lambda = \sum_i \lambda_i, N = \sum_i N_i$, and thus fractional parameters $(x,y) = \left(\frac{\sum_i \lambda_i}{r}, \frac{\sum_i N_i}{r}\right) = \frac{1}{r}\left(\sum_i r_i x_i, \sum_i r_i y_i\right) = \frac{\sum_i r_i (x_i, y_i)}{r}$. That is, (x, y) is a convex linear combination of the (x_i, y_i). Therefore, the region containing the possible fractional parameters is a convex set in the (x, y) plane. Since (as in [11]) the $[r, \lambda]$-colourings of G may be characterised as the feasible solutions of an integer programming problem, this convex set is the linear image of a convex polytope, and is therefore a polygon. We call this polygon the *chromatic polygon* CP(G) of G. (To be precise, the *rational* pairs (x, y) in CP(G) are the possible parameter pairs of colourings of G.)

General properties of CP(G) for any graph G

We first find the upper boundary of CP(G) by means of the following lemma.

Lemma 7.1 Let G be any connected graph. Then the maximum number of colours that can be used in any $[r, \lambda]$-colouring of G is $r + (n-1)(r-\lambda) = nr - (n-1)\lambda$.

Proof Order the vertices $v_0, \ldots$, such that vertex v_i $(i \geq 1)$ is adjacent to at least one vertex v_j $(j < i)$. Now colour the vertices in order. We must allocate r colours to vertex v_0; each subsequent vertex can be allocated at most $r - \lambda$ *new* colours. Thus, *at most* $r + (n-1)(r-\lambda) = nr - (n-1)\lambda$ colours can be used. This number can be achieved by allocating a set of λ colours common to all vertices, then adding disjoint sets of $r-\lambda$ further colours to each vertex. ■

Since a disconnected graph can be coloured by allocating disjoint sets of colours to distinct components, we have the following corollary.

Corollary 7.2 Let G be any graph with k components. Then the maximum number of colours in an $[r,\lambda]$-colouring of G is $kr+(n-k)(r-\lambda)=nr-(n-k)\lambda$. ■

Proposition 7.3 The upper boundary of CP(G) for any graph G with n vertices and k components is the line $y=k+(n-k)(1-x)=n-(n-k)x$ $(0\le x\le 1)$, while the lower boundary is the Graph of $y=\chi_f[x](G)$ $(0\le x\le 1)$.

Proof The description of the upper boundary follows from Corollary 7.2, while that of the lower boundary follows from the definition of $\chi_f[x](G)$ given in Chapter 1.

Corollary 7.4 Let G be any non-null graph. There are no points of CP(G) below the line $x+y=2$.

Proof Since $\chi_{r,\lambda}(G)\ge 2r-\lambda$, by Proposition 1.1, the result is immediate.

Complementation in the Chromatic Polygon

A colour used in a colouring is *universal* if it occurs on every vertex, and a colouring is *universal-free* if it has no such colours. We recall from Chapter 1 that the *complementary colouring* of a colouring μ is the colouring in which each vertex receives exactly those colours that it did not receive in μ.

If μ is universal-free, the parameters of μ^C are $[N-r,N-2r+\lambda,N]$, and so μ^C has fractional parameters $\left(\frac{N-2r+\lambda}{N-r},\frac{N}{N-r}\right)=\left(\frac{x+y-2}{y-1},\frac{y}{y-1}\right)$. Every juxtaposition of universal-free colourings also has this property, and so there is a subpolygon SP(G) of CP(G) that corresponds to all the universal-free overlap colourings.

Proposition 7.5 SP(G) is invariant under the transformation

$$\tau(x,y)\to\left(\frac{x+y-2}{y-1},\frac{y}{y-1}\right).$$

Proof The operation of complementation maps the set of universal-free colourings onto itself, and the effect of this operation on the fractional parameters is exactly τ. ■

Corollary 7.6 The point (0, 2), which occurs only if G is bipartite, is self-complementary.

Proof Substituting in A, $(0,2) \mapsto \left(\frac{0+2-2+0}{1}, \frac{2}{1}\right) = (0,2)$. ■

Corollary 7.7 Any point on the y-axis is mapped to a point on the line L, $x+y=2$. ■

Proof For any point on the y-axis, $\left(0, \frac{N}{r}\right) \mapsto \left(\frac{N-2r}{N-r}, \frac{N}{N-r}\right)$, which lies on L.

Corollary 7.8 Any point on L, with the exception of (1, 1), is mapped to a point on the y-axis.

Proof For any point on L (except (1, 1)),

$$\left(\tfrac{\lambda}{r}, 2-\tfrac{\lambda}{r}\right) \mapsto \left(\frac{2-\frac{\lambda}{r}+\frac{\lambda}{r}-2}{1-\frac{\lambda}{r}}, \frac{2-\frac{\lambda}{r}}{1-\frac{\lambda}{r}}\right) = \left(\frac{0}{r-\lambda}, \frac{2r-\lambda}{r-\lambda}\right).$$

This is a point on the y-axis, except for the trivial case $r=\lambda$, giving an indeterminate expression. ■

Chromatic Polygons of Cycle Graphs, Wheels and Complete Graphs

Since the cycle graphs, wheels and complete graphs are connected, Proposition 7.3 implies that the upper boundary of CP(G) for any of these graphs is the Graph of $y = n-(n-1)x$ $(0 \le x \le 1)$ while the lower boundaries are given by the Graph of $\chi_f[x](G)$ for each of these types.

Theorem 7.9 (i) $\chi_f[x](C_{2p}) = 2-x$ $(0 \le x \le 1)$.

(ii) $$\chi_f[x](C_{2p+1}) = \begin{cases} (2+\frac{1}{p})(1-x) & 0 \le x \le \frac{1}{p+1} \\ 2-x & \frac{1}{p+1} \le x \le 1 \end{cases}$$

Proof Part (i) follows from Proposition 1.3, since C_{2p} is bipartite, while Part (ii) follows from Theorem 2.1 (since $(2+\frac{1}{p})(1-x) \ge 2-x$ when $0 \le x \le \frac{1}{p+1}$) and $(2+\frac{1}{p})(1-x) \le 2-x$ when $\frac{1}{p+1} \le x \le 1$). ■

Theorem 7.10 (i) $\chi_f[x](W_{2p+1}) = \begin{cases} 3(1-x) & 0 \le x \le \frac{1}{2} \\ 2-x & \frac{1}{2} \le x \le 1 \end{cases}$

(ii) $\chi_f[x](W_{2p+2}) = \begin{cases} 3-4x+\dfrac{1-2x}{p} & 0 \le x \le \dfrac{1}{p+2} \\ 3(1-x) & \dfrac{1}{p+2} \le x \le \dfrac{p}{2p+1} \\ 3-2x-\dfrac{p}{2p+1} & \dfrac{p}{2p+1} \le x \le \dfrac{p+1}{2p+1} \\ 2-x & \dfrac{p+1}{2p+1} \le x \le 1 \end{cases}$

These follow from Proposition 3.2 and Theorem 3.10, respectively. ■

Annex 7.1 displays the chromatic polygons of the wheel graphs W_4, W_6, W_8 and W_{10}.

Common Points of CP(K_n) and CP(K_{n+1})

Lemma 7.11 For every vertex v_a of CP(K_n), $(x,y) = \left(\frac{a-1}{n-1}, \frac{n}{a}\right)$ $(a \in [1, ..., n])$.

Proof Corollary 6.4 shows that the point $(x,y) = \left(\frac{\lambda}{r}, \frac{N}{r}\right)$ lies on the hyperbola $y = \frac{n}{(n-1)x+1}$ for an equable colouring of K_n and strictly above this hyperbola for a non-equable colouring. The only possible values of f for an equable colouring of K_n are the integers 1, ..., n; moreover, by Proposition 6.8, for each $a = 1, ..., n$ there is an equable colouring of K_n with $f = a$, and with $\frac{\lambda}{r} = \frac{a-1}{n-1}$ and $\frac{N}{r} = \frac{n}{a}$ (as may be seen either from the comparison with design theory on page 41 or from the fact that here,

$$\frac{\lambda}{r} = \frac{\binom{n-2}{a-2}}{\binom{n-1}{a-1}} \text{ and } \frac{N}{r} = \frac{\binom{n}{a}}{\binom{n-1}{a-1}}).$$

Thus, every vertex of CP(K_n) must lie on the hyperbola $y = \frac{n}{(n-1)x+1}$ and the values $a = 1, \dots, n$ correspond precisely to these vertices. ■

Corollary 7.12 For all n, vertex v_{n-1} of CP(K_n) lies on the line $x + y = 2$.

Proof The vertex v_{n-1} is the point $\left(\frac{n-2}{n-1}, \frac{n}{n-1}\right)$. ■

Theorem 7.13 Every vertex of CP(K_{n+1}) lies on an edge of CP(K_n).

Proof Denote the n vertices of CP(K_n) by v_1 (i e $(0, n)$), ..., v_n (i e $(1, 1)$).

Then v_a of $CP(K_n)$ is the point $\left(\frac{a-1}{n-1}, \frac{n}{a}\right)$; v_{a+1} of $CP(K_n)$ is the point $\left(\frac{a}{n-1}, \frac{n}{a+1}\right)$.

The equation of the edge $v_a v_{a+1}$ is therefore $\dfrac{y-\frac{n}{a+1}}{\frac{n}{a}-\frac{n}{a+1}} = \dfrac{x-\frac{a}{n-1}}{\frac{a-1}{n-1}-\frac{a}{n-1}}$,

which simplifies to $$a(a+1)y = 2an - n(n-1)x.$$

Now, vertex v_{a+1} of $CP(K_{n+1})$ is the point $\left(\frac{a}{n}, \frac{n+1}{a+1}\right)$.

Substituting $x = \frac{a}{n}$ in B, we have:

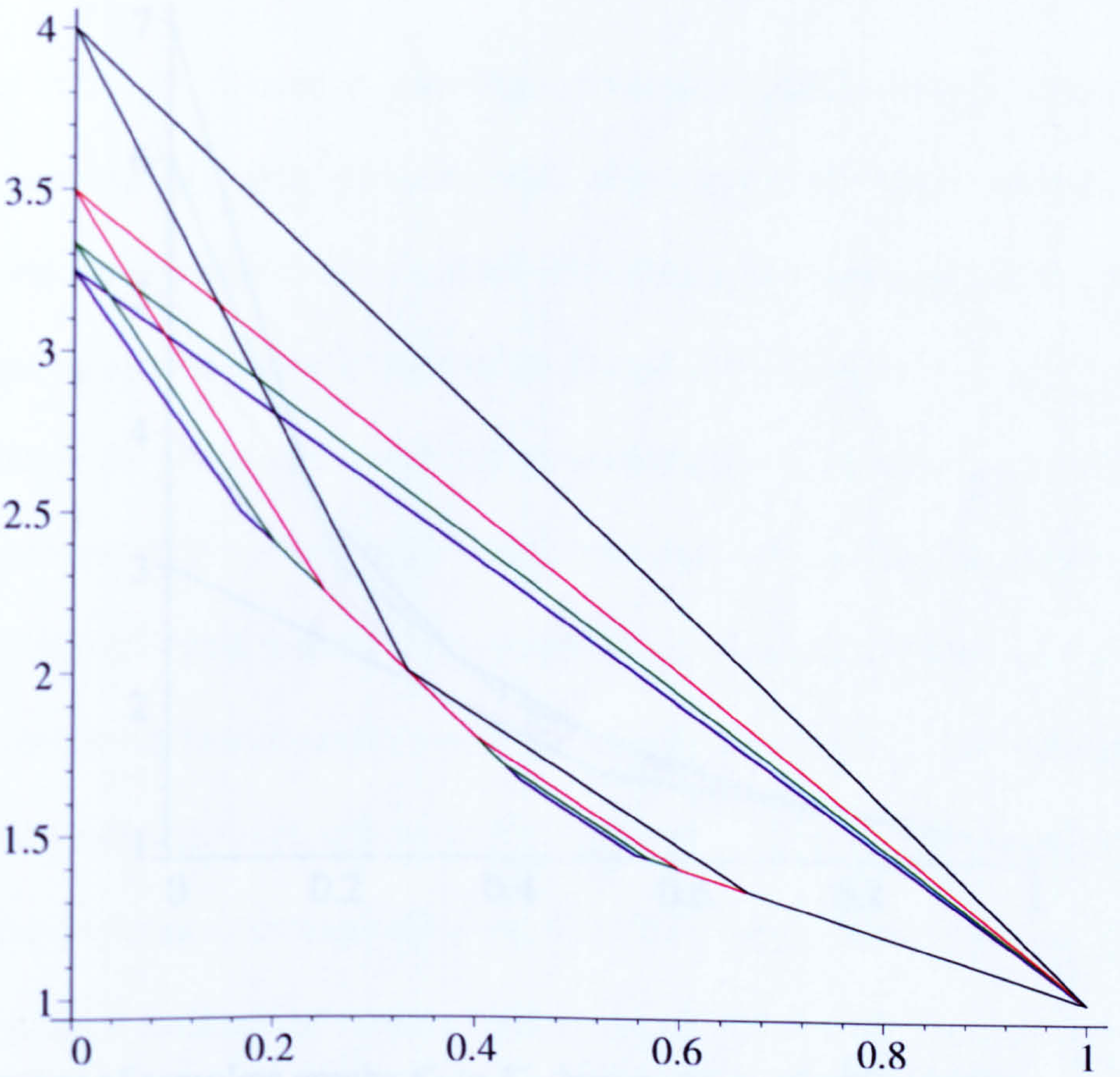
$$a(a+1)y = 2an - a(n-1), \text{ i e } y = \frac{n+1}{a+1}, \text{ and, since } \frac{a-1}{n-1} < \frac{a}{n} < \frac{a}{n-1},$$

vertex v_{a+1} of $CP(K_{n+1})$ lies on edge $v_a v_{a+1}$ of $CP(K_n)$. ■

Annex 7.2 shows the above for $CP(K_3)$ - $CP(K_7)$.

Annex 7.1

Chromatic Polygons of Wheel Graphs

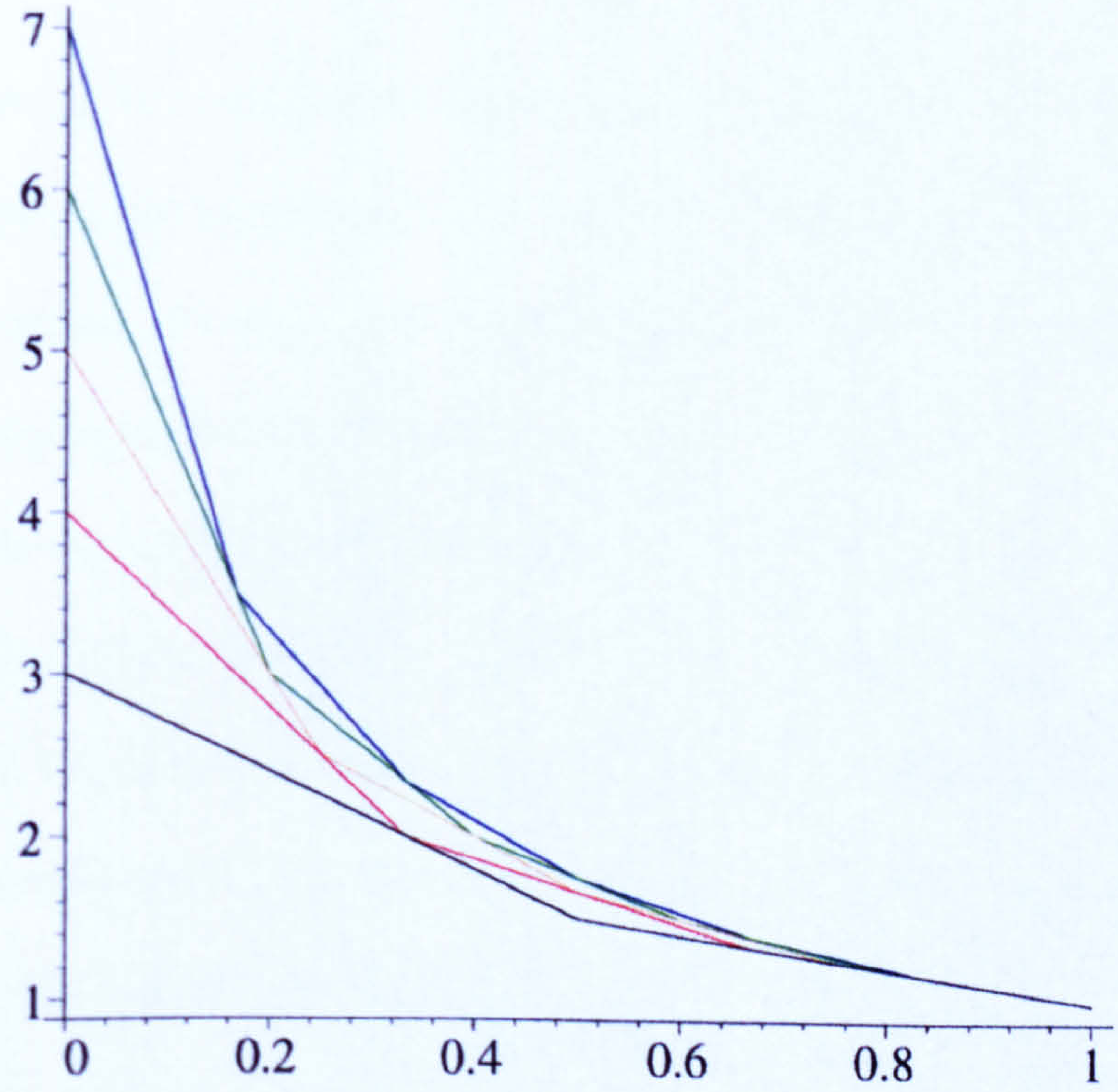


Chromatic polygons of even wheel graphs W_4 to W_{10}, coloured:

W_4	Black
W_6	Red
W_8	Green
W_{10}	Blue

Annex 7.2

Common Points of Chromatic Polygons of K_n and K_{n-1}



Chromatic polygons of complete graphs K_3 to K_7, lower edges only (for clarity), coloured:

K_3	Black
K_4	Red
K_5	Gold
K_6	Green
K_7	Blue

Chapter 8: Overlap Colourings and Statistical Applications

This chapter shows that certain designs for statistical experiments turn out, on examination, to be overlap colourings of graphs other than complete graphs.

As we saw in Chapter 6, overlap colourings of complete graphs correspond closely with BIBDs; in an equable overlap colouring of K_n, if we interpret the vertex set as a set of varieties and the palette of colours as a set of blocks, then we arrive at a BIBD; the set of vertices having a given colour is regarded as a block of the design.

Now BIBDs in the design of experiments are used in order to allocate experiments in a situation where it is impracticable for every experiment to be performed on every available subject. Typically (see [18]) one may have v varieties of wheat and b physically different sites, so that ideally every variety should be grown at every site, but this may be economically prohibitive (or, indeed the sites may not be large enough). A BIBD would then allocate varieties to sites in such a way that each wheat variety is grown at the same number r of sites, each pair of varieties being grown together at the same number λ of sites. The required layout is then a BIBD, corresponding to an $[r, \lambda]$ colouring of K_n.

However, there are many types of experimental design that are more subtle than the above. Bailey [3] defines *orthogonal block structures*, in which a set of experimental plots is subjected to two or more *orthogonal uniform partitions*. That is, If P is the set of all the plots and $Q_1, \ldots, Q_q$ the set of partitions, then each partition is into subsets of P of equal size, and the subsets of P arising from applying simultaneously any two partitions Q_i, Q_j are again of equal size. (For example, the plots may be laid out in a rectangular array, so that they are partitioned into rows of equal size and also into columns of equal size).

Designs arising from such partitions are known as *nested block*, or *split-block, designs*.

Donald Preece (personal communication) has given a simple such example, in which

18 plots are divided into 9 blocks of size 2, laid out in a square array, so that there are two types of blocks of 6 plots, corresponding to rows and columns. In Preece's example (below), varieties A, B, C are allocated to the plots in such a way that each row, and each column, forms a BIBD with parameters (3, 3, 2, 2, 1) and hence a [2, 1] colouring of K_3 with $N = 3$.

$A\ B$	$B\ C$	$C\ A$
$B\ C$	$C\ A$	$A\ B$
$C\ A$	$A\ B$	$B\ C$

The design as a whole, then, corresponds to a colouring of all 9 blocks of size 2, in which any two blocks in the same row overlap by 1 as do any two blocks in the same column. That is, it is a [2, 1] colouring of the Cartesian product $K_3 \square K_3$. where (see [13]) the *Cartesian product* $G \square H$ of two graphs G and H has the usual Cartesian product $V(G) \times V(H)$ as the vertex set, where (v_1, w_1) is adjacent to (v_2, w_2) if and only if *either*

$v_1 = v_2$ and w_1 is adjacent to w_2 *or*

$w_1 = w_2$ and v_1 is adjacent to v_2.

Bailey [3] gives several examples of split-block designs, but usually the 'component' designs are (0, 1)-designs, that is, each overlap is either 1 or 2. Below, we give an example of a split-block design in which there is an overlap of 1 between any two blocks in the same row and also between any two blocks in the same column.

1 2 4	2 3 5	3 4 6	4 5 7	5 6 1	6 7 2	7 1 3
2 3 5	3 4 6	4 5 7	5 6 1	6 7 2	7 1 3	1 2 4
3 4 6	4 5 7	5 6 1	6 7 2	7 1 3	1 2 4	2 3 5

This then is a [3, 1] colouring of $K_3 \square K_7$ with $N = 7$.

Note that $N = \max\{\chi_{3,1}(K_3), \chi_{3,1}(K_7)\}$. This exemplifies the following general property.

Theorem 8.1 Let $G = K^{(1)} \square K^{(2)} \square \ldots \square K^{(m)}$ be a Cartesian product of complete graphs, where each $K^{(i)}$ has order n_i and $n_1 \le n_2 \le \ldots \le n_m = n$. Then $\chi_{r,\lambda}(G) = \chi_{r,\lambda}(K_n)$.

Proof For each $i = 1, \ldots, m$ let us identify $V(K^{(i)})$ with the elements $0, 1, \ldots, n_i - 1$ of the cyclic group of order n. Let f be the following function from $V(G)$ to $V(K_n)$:

$$f(z_1, \ldots, z_m) = \sum_{i=1}^{m} z_i$$

(with addition modulo n). If $(y_1, \ldots, y_m)$ and $(z_1, \ldots, z_m)$ are adjacent in G, then $y_i \neq z_i$ for exactly one value of i, and so $f(y_1, \ldots, y_m) \neq f(z_1, \ldots, z_m)$. Thus $f(y_1, \ldots, y_m)$ and $f(z_1, \ldots, z_m)$ are adjacent. Hence, f is a a homomorphism from G to K_n, and the result follows from Proposition 1.2 (since G contains K_n as a subgraph). ■

In terms of split-block designs, this theorem states that the number of varieties in a split-block design is the largest number of varieties in the BIBDs corresponding to the individual partitions.

However, not all experimental designs need be based on complete graphs in this way. We may be particularly interested in comparisons involving 'neighbouring' blocks in some sense (that is, the blocks in each partition may form a graph other than a complete graph), so that the corresponding split-plot design may correspond to a Cartesian product of graphs other than complete graphs. For example, if the blocks have a cyclic adjacency structure, then a split-plot design corresponding to a Cartesian product of cycles may be appropriate.

It is an open question whether Theorem 8.1 generalizes to arbitrary Cartesian products of graphs, but it does generalize to products of cycles, as we now show.

Theorem 8.2 Let $G = C^{(1)} \square C^{(m)}$ be a Cartesian product of cycles, where each $C^{(i)}$ has order n_i, and if any n_i is odd then $C^{(m)} = C_{2p+1}$, the smallest odd cycle. Then

$$\chi_{r,\lambda}(G) = \chi_{r,\lambda}(C^{(m)}).$$

Proof If all the n_i are even, then G is bipartite and $\chi_{r,\lambda}(G) = 2r - \lambda$ by Proposition 1.3.

Otherwise, for each $i = 1, \ldots, m$, identify $V(C^{(i)})$ with the elements $0, 1, \ldots, n-1$.

We define the following *roll functions*:

$$\text{roll}_i\colon \{0,1,\ldots,n_i-1\} \to Z_{2p+1}\ (i = 1, \ldots, n)\text{:}$$

If n_i is even, then $\text{roll}_i(z) = \left\{ \begin{array}{ll} 0, & z \text{ even} \\ 1, & z \text{ odd} \end{array} \right\}$

If $n_i = 2p + 1$, then $\text{roll}_i(z) = z$ ($z = 1, 2, \ldots, 2p$)

If $n_i = 2(p + s) + 1$ (where $s > 0$),

then $\text{roll}_i(z) = \left\{ \begin{array}{ll} 0, & z = 0,2,\ldots,2s \\ 1, & z = 1,3,\ldots,2s+1 \\ z-2s, & z = 2s+2, 2s+3, \ldots, 2(p+s) \end{array} \right\}$

Next, we define the function ROLL: $Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{2p+1} \to Z_{2p+1}$, as follows:

Let $\text{z} = (z_1, z_2, \ldots, z_n)$; then

$$\text{ROLL(z)} = \sum_{i=1}^{m} \text{roll}_i(z_i)$$

As in the proof of Theorem 8.1, if $(y_1, \ldots, y_m)$ and $(z_1, \ldots, z_m)$ are adjacent in G, then $f(y_1, \ldots, y_m)$ and $f(z_1, \ldots, z_m)$ are adjacent. Thus, f is a homomorphism from G to C_{2p+1}, and the result follows as in Theorem 8.1. ∎

Conclusion

This thesis adds to the literature on variants of graph colouring theory, by introducing the parameter $\chi_{r,\lambda}(G)$ (the number of colours required to colour each vertex of G with r colours with an overlap of λ between adjacent vertices), and also the fractional version, $\chi_f[x](G)$, being the least value attained over all r by $\frac{\chi_{r,xr}(G)}{r}$ (for $0 \le x \le 1$).

After some fundamentals concerning these parameters, they are investigated for a number of classes of graphs. Brief consideration is given to their relations to more well-known graph-theoretic parameters; and the fractional parameters are shown to be expressible in terms of the 'chromatic polygon'. The work of the thesis is related to design theory, codes and statistical designs.

Fundamentals

Chapter 1 defines the basic concepts and gives some fundamental properties. In particular, $\chi_{r,\lambda}(G) \ge 2r - \lambda$ for any non-null graph, the bound being attained for bipartite graphs (page 10); and when a graph has a vertex that is adjacent to all the others, then any equable $[r, \lambda]$ colouring (that is, all colours occurring equally often) uses exactly $\frac{nr^2}{(n-1)\lambda + r}$ colours (page 12).

Overlap parameters

Chapters 2, 3, 5 and 6 deal with cycles, wheels, generalized Petersen graphs and complete graphs, respectively. All the overlap parameters of cycles and wheels are obtained (see pages 17, 29 and 38), as are the fractional (non-overlap) chromatic numbers of the generalized Petersen graphs (see page 58 for a general result). Partial results are obtained concerning the complete graphs; the most significant of these (page 70) gives the correct overlap chromatic number provided there is an equable colouring (and hence gives $\chi_f[x](K_n), 0 \le x \le 1,$ as is shown in Chapter 7).

Relation with other graph parameters

Chapter 2 shows that for every relevant r, λ, p there is an $[r, \lambda]$ colouring of C_{2p+1} (using $\chi_{r,\lambda}(C_{2p+1})$ colours) that is a juxtaposition of a small number of *primitive* colourings (page 24). This leads to the work of Chapter 4, which investigates the place of overlap chromatic numbers in the classification of graphs. The *core* of a graph G is the smallest subgraph to which G has a homomorphism, and has the same overlap profile as G; thus, classification by cores is at least as fine as classification by their overlap profiles.

It is shown (page 40) that any graph with the same multichromatic profile as an odd cycle also has the same overlap profile as that cycle. This leads to the result (page 42) that the *bangle* $B(2q + 1, 2p + 1)$ also has the same overlap profile as C_{2p+1}. However, there is no homomorphism from $B(3, 2p + 1)$ to C_{2p+1}, and so classification by core is strictly finer than by overlap profile.

Classification by overlap profile is clearly at least as fine as by multichromatic profile. However, the result quoted above (page 40) shows that in the case of graphs with the multichromatic profile of an odd cycle, the classifications are the same.

The chromatic polygon

Chapter 7 discusses the general properties of the chromatic polygon of a graph G, namely the plane polygon within which the point $\left(\frac{\lambda}{r}, \frac{N}{r}\right)$ must lie for any $[r, \lambda]$ colouring of G using N colours (pages 82 - 84). Precise descriptions are given of the chromatic polygons of the cycles, wheels and complete graphs.

Relevance to design (including statistical design) theory and coding theory

The complete graphs considered in Chapter 6 are closely connected both with BIBDs and with constant-weight codes, and the chapter explores these connections, showing in particular that upper bounds on lengths of constant-weight codes yield lower bounds on overlap chromatic numbers of complete graphs, and conversely, upper bounds on the overlap parameters for complete graphs imply lower bounds for lengths of constant-weight codes. (See pages 77, 78.) The chapter concludes (page 79) with a table illustrating lower and upper bounds for some overlap parameters of some complete graphs.

Finally, Chapter 8 broadens the discussion of the relationship between overlap colourings and BIBDs, by considering more general experimental designs - in particular, those in which the experimental sites are partitioned in ways that seem to ask for the overlap parameters of Cartesian products of complete graphs and of cycles.

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